Mathematics 251: Lab 2   QUADRIC SURFACES

A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general form of such an equation is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, D, E, F, G, H, I,$ and J are constants. In this lab, we use Maple to help visualize some types of quadric surfaces that can arise from equations of the above form.

Please turn in only the printout of your Maple worksheet, which should include the Maple commands you input and Maple's response. Your worksheet should also include explicit answers to all questions asked and labels on any plots included in your worksheet. These should be inserted by using the **text** feature of Maple. Also use the **text** feature of Maple to include your name and section number before each problem (do NOT write any of this material in by hand). Be sure to remove from the worksheet any extraneous material and any errors you have made.

Before you start up Maple, first copy the file lab2.mws into your directory from the Web page of the course. After you start up Maple, import this file into Maple by choosing **Open** from the **File** menu as described in the handout **Instructions for Use of Maple in Mathematics 251**. You will be asked to execute the Maple commands that have been placed in this worksheet and to add some of your own.

There are three Maple commands that we will use to plot surfaces in three dimensions. These are `plot3d`, `implicitplot3d`, and `display`. Before the second or third of these can be used, one must type the command `with(plots):`. The command `plot3d` may be used to graph a surface when one of the variables can be solved for explicitly in terms of the other two, e.g., $z = f(x, y)$, while the command `implicitplot3d` may be used to graph any surface of the form $g(x, y, z) = 0$. Note that by writing $g(x, y, z) = z - f(x, y)$, any surface of the first form is also of the second form, so the `implicitplot3d` command is more general. The command `display` is used to display two previously defined plots at the same time on the same set of axes.

In this lab, we are interested in viewing the intersection of two surfaces. Although it is possible to use the syntax `plot3d({x^2 +y^2,4}, x=0..1,y=0..1);` to graph the two surfaces $z = x^2 + y^2$ and $z = 4$ on the same plot, we shall adopt an alternative approach using the `display` command which allows us to choose a different color for each plot so that we can more easily see the curve of intersection of the two surfaces. To use the `display` command to graph two surfaces on the same plot, one gives names to each of the plots (e.g., `p1` and `p2`) and uses the command `display({p1,p2});`. Note that to simultaneously display two surfaces on the same plot which have been generated using the `plot3d` command, they must both be solved for the same variable in terms of the other two. Otherwise, the `implicitplot3d` command should be used to generate the surfaces. These ideas are illustrated in the following sequence of commands, which should appear at the beginning of Maple worksheet lab2.mws. Execute them to verify that the first sequence using the `plot3d` command does not give a correct simultaneous plot of the planes $x = 1$

and $z = 2$, while the second sequence using the `implicitplot3d` command does.

```
with(plots):
p1:=plot3d(1, y=0..1,z=0..2, color=RED):
p2:=plot3d(2, x=0..1,y=0..1, color=BLUE):
display({p1,p2});

p3:=implicitplot3d(x=1, x=0..1, y=0..1,z=0..2, color=RED):
p4:=implicitplot3d(z=2, x=0..1,y=0..1,z=0..2, color=BLUE):
display({p3,p4});
```

One problem with the above sequence of commands is that one does not see any of the plots until the `display` command is used. A better way to proceed is to first type

```
plot3d(1, y=0..1,z=0..2, color=RED);
```

and view the plot. If it is satisfactory, edit the line by inserting `p1:=` at the beginning and changing the semicolon at the end to a colon, and then execute it again. Note that the only effect of not changing the semicolon to a colon is to cause Maple to write some (usually unwanted) output to the screen. Using the colon suppresses this output.

1a. An ellipsoid is a quadric surface of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad (*)$$

where a, b, and c are positive constants.

One approach to plotting such a surface is to solve for z, obtaining the two surfaces

$$z = c\sqrt{1 - x^2/a^2 - y^2/b^2}, \qquad z = -c\sqrt{1 - x^2/a^2 - y^2/b^2}$$

which correspond to the top and bottom half of the ellipsoid. Each of these surfaces is defined explicitly in the form $z = f(x, y)$ and thus can be graphed using the `plot3d` command. The other approach is to use the original equations which define the surface implicitly and use the `implicitplot3d` command.

Note that when either of these commands is used, you must specify the domain of the variables (x and y when plotting $z = f(x, y)$ using the `plot3d` command and x, y, and z, when plotting $g(x, y, z) = 0$ using the `implicitplot3d` command). In the case of the top and bottom halves of the ellipsoid above, the argument of the square root must be ≥ 0. This may be accomplished by letting $-a \leq x \leq a$ and then $-b\sqrt{1 - (x^2/a^2)} \leq y \leq b\sqrt{1 - (x^2/a^2)}$. If you specify the domain as, for example, $-a \leq x \leq a$, $-b \leq y \leq b$, Maple will still produce a plot, ignoring the values for which z would be imaginary. However, the plot may not be as precise. When plotting $g(x, y, z) = 0$ using the `implicitplot3d` command, you must specify maximum ranges for x, y, and z in terms of real numbers. Maple plots the values which satisfy the equation.

In this problem, we consider the ellipsoid with $a = 2$, $b = 3$, $c = 5$. Execute the sequence of commands listed under Problem 1 in Maple worksheet lab2.mws to see how Maple can produce plots of:

(i) the top half of the ellipsoid using the `plot3d` command and the range $-a \le x \le a$, $-b \le y \le b$,

(ii) the top half of the ellipsoid using the `plot3d` command and the range $-a \le x \le a$, $-b\sqrt{1-(x^2/a^2)} \le y \le b\sqrt{1-(x^2/a^2)}$,

(iii) both halves of the ellipsoid using the `plot3d` command,

(iv) both halves of the ellipsoid using the `implicitplot3d` command,

(v) the ellipsoid and the plane $x = 1$ using the `implicitplot3d` command, with different colors assigned to each plot,

Note that the options `axes=BOXED` and `scaling=CONSTRAINED` have been incorporated directly into each of the above plots. They can also be added or deleted from an already existing plot by choosing from the **Axes** and **Projection** menus. Thus, you can easily see what happens when they are deleted. The default resolution (i.e., what Maple assumes if nothing is specified) is grid=[10,10,10]. Since increasing the resolution can considerably slow down the execution of a plot and take up a considerable amount of disk space (especially a three-dimensional plot), be careful not to choose too large a value. With the exception of Problem 1, the default resolution should be sufficient for all the problems in this assignment.

1b. Plot the ellipsoid and the plane $y = 2$ on the same set of axes for $-2 \le x \le 2$, $-3 \le y \le 3$, and $-5 \le z \le 5$ using the `implicitplot3d` command with two colors, `axes=BOXED`, `scaling=CONSTRAINED`, and `grid=[15,15,15]`. Rotate the plot until it clearly shows the curve of intersection.

1c. Repeat (1b) with the plane $y = 2$ replaced by the plane $z = 3$.

1d. Use the **text** feature of Maple to insert the equation (you don't need Maple to find it) and type of the curve of intersection of the ellipsoid with the plane $y = 2$.

The next 3 problems can be done using commands that are very similar to those used in Problem 1. To avoid problems with lack of memory and *Maple* files which are too large, use the default resolution (grid=[10,10,10]) and do each problem in a separate worksheet with a different name. So **SAVE** your worksheet containing Problem 1, **CLOSE** the file and **OPEN** a new file with a different name.

2. The surface

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

is called an elliptic paraboloid.

2a. Plot the elliptic paraboloid corresponding to $a = 2$, $b = 3$, $c = 5$ over the domain $-2 \le x \le 2$, $-3 \le y \le 3$. Note that z will then range over the interval $[0, 10]$.

2b. Obtain three plots which clearly show the intersection of this paraboloid with the planes $x = 1$, $y = 2$, $z = 3$, respectively.

2c. Determine without the use of Maple the equation of the curve of intersection for each of the planes and use the **text** feature of Maple to enter it into your Maple worksheet, making sure to label your equation (e.g., Equation of curve of intersection (x=1)). Also state explicitly what type of curve this is.

3. The surface

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

is called a hyperbolic paraboloid.

3a. Plot the hyperbolic paraboloid corresponding to $a = 2$, $b = 3$, $c = 5$ over the domain $-2 \leq x \leq 2$, $-3 \leq y \leq 3$.

3b. Obtain three plots which clearly show the intersection of this paraboloid with the planes $x = 1$, $y = 2$, $z = 3$, respectively.

3c. Determine without the use of Maple the equation of the curve of intersection for each of the planes and use the **text** feature of Maple to enter it into your Maple worksheet, making sure to label your equation (e.g., Equation of curve of intersection (x=1)). Also state explicitly what type of curve this is.

4. Consider the family of surfaces given by

$$z = x^2 + y^2 + dxy$$

for different values of the constant d. Although these surfaces are not exactly of the form described in the previous problems, for some values of the constant d they are elliptic paraboloids and for others they are hyperbolic paraboloids. In this problem we try to determine a transitional value of d for which the surface changes from one type to another by viewing the graphs of this surface for various values of the constant d.

4a. Plot the surfaces corresponding to the values $d = 0, 1, 2, 3, 4$ over the range $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. Then rotate each surface until you can identify its type.

4b. From among these surfaces, choose one which is an elliptic paraboloid, labeling it as elliptic and with the value of d to which it corresponds.

4c. From among these surfaces, choose one which is a hyperbolic paraboloid, labeling it as hyperbolic and with the value of d to which it corresponds.

4d. From among these surfaces, choose one which is a transitional surface, labeling it as transitional and with the value of d to which it corresponds.

4e. Remove the two surfaces you have not labeled from your Maple worksheet.