Math 251:05–06 — Spring 2003
Lecture digest for third exam

Lecture #12

Examples of Surfaces

Section 12.6 section describes a few simple examples of surfaces. Generally, surfaces will be given by an equation $g(x, y, z) = 0$. The surfaces considered are (general) cylinders and cones, and surfaces given by equations of degree s, which are called **quadrics**. Although the examples of quadrics are in special position, they include all types of surfaces.

Cylinders

An equation in which one of the variables does not appear describes a **cylinder**. All values of the missing variable are allowed while the included variables describe a curve in one of the coordinate planes that gives the common cross-section. Note that the shape of the cross-section is arbitrary.

Cones

Another way to trace a surface with lines is to take a curve in the plane $z = 1$ and join all points on the curve to the origin. Then a point (x, y, z) with $z \neq 0$ lies on this surface if and only if $(x/z, y/z, 1)$ lies on the given curve. If the curve has a polynomial equation, it can be multiplied by a power of z to obtain a polynomial equation of the cone. This equation has the special property that all terms have the same degree.

Conversely, all such **homogeneous** equations define cones.

Quadrics

A surface with an equation of second degree is called a **quadric**. Several such surfaces are explored in Maple Lab 2. Only the special cases with equations

$$Ax^2 + By^2 + Cz^2 + J = 0$$

or

$$Ax^2 + By^2 + z = 0$$

will be considered here since the general equation can be put in one of these forms by a rotation and translation of coordinates.

A major tool in recognizing these surfaces is to consider the intersections with the coordinate planes.

Figures of Rotation

If $A = B$, the sections in the xz plane and the yz plane look the same. Indeed, one has the same section in any plane through the z axis. If we denote the distance from the z axis by r, then $r^2 = x^2 + y^2$ and the equation depends only on r and z. This r is the r of **polar coordinates** in the xy plane. More generally, we can use polar coordinates r and θ as an alternative to the rectangular coordinates x and y, keeping a z coordinate with its usual interpretation. This gives the **cylindrical coordinates** in space that will be considered later in this course.

Changes of Scale

If x is replaced by cx for some constant c without changing the other coordinates, the effect may be described by changing the scale on the x axis without changing the graph. If you prefer to have fixed scales on the axes, this corresponds to shrinking or stretching uniformly in a direction parallel to the x axis. Such transformations take circles into ellipses.

Graphs of Functions

One of the easiest descriptions of surfaces is $z = f(x, y)$. The **implicit function theorem** says that our general formula $g(x, y, z) = 0$ may be assumed to be of this special form in a neighborhood of any point with $g_3(x, y, z) \neq 0$. Here g_3 refers to the partial derivative with respect to the third variable.

Some examples

#9. $x^2 - y^2 + z^2 = 1$.
#11. $4x^2 + 9y^2 + 26z^2 = 36$.
#15. $y^2 = x^2 + z^2$.
#31. $z = x^2 + y^2 + 1$.

Using Green's Theorem for change of coordinates, part 1

The proof of Green's theorem shows that every double integral in the xy can be evaluated as a suitable line integral around its boundary. If x and y are functions of u and v, this is converted to a line integral in the uv plane. This line integral is converted by Green's theorem into a double integral in the uv plane. So far, we have done this only for polar coordinates, and no separate r, θ plane, in which these are **rectangular** coordinates was introduced. However, the proof via Green's Theorem needs this plane for a **second** application of Green's Theorem. In this proof, we can start with a line integral

$$I = \oint P\,dx + Q\,dy.$$

Here, P and Q will be functions of x and y. We then express x and y in terms of u and v, which will be expressed in terms of u and v by composition with the functions giving x and y in terms of u and v. This substitution will express I as a line integral in the uv-plane. In addition to obtaining P and Q in terms of u and v, we need to write

$$\begin{aligned} dx &= x_u\,du + x_v\,dv, \\ dy &= y_u\,du + y_v\,dv. \end{aligned}$$

Using Green's Theorem for change of coordinates, part 2

Thus,

$$I = \oint (Px_u + Qy_u)\,du + (Px_v + Qy_v)\,dv.$$

Applying Green's theorem in the uv-plane gives an integrand of the double integral that is

$$\begin{aligned} &(Px_v + Qy_v)_u - (Px_u + Qy_u)_v = \\ &\quad Px_{vu} + Qy_{vu} + (P_x x_u + P_y y_u)x_v \\ &\qquad\qquad + (Q_x x_u + Q_y y_u)y_v \\ &\quad -Px_{uv} - Qy_{uv} - (P_x x_v + P_y y_v)x_u \\ &\qquad\qquad - (Q_x x_v + Q_y y_v)y_u \\ &\quad = (Q_x - P_y)(x_u y_v - x_v y_u) \end{aligned}$$

Here the first part is the integrand of the original double integral in the xy plane composed with the definition of x and y in terms of u and v, and the second factor is the ratio of areas in the two planes. In polar coordinates, this reduces to r.

Lecture #13

Area of parallelograms, part 1

We noted in lecture 7 that The length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram with **a** and **b** as sides. We should take a closer look.

Suppose we have vectors

$$\mathbf{v}_0 = \langle a_0, b_0, c_0 \rangle \quad \mathbf{v}_1 = \langle a_1, b_1, c_1 \rangle$$

based at a point $P(x_0, y_0, z_0)$ in $\mathbb{R}^3$. Then the four points P, $P + \mathbf{v}_0$, $P + \mathbf{v}_1$, $P + \mathbf{v}_0 + \mathbf{v}_1$ are the vertices of a parallelogram $\mathcal{P}$ in space, which lies in the plane

$$Ax + By + Cz = D$$

where

$$\langle A, B, C\rangle = \mathbf{v}_0 \times \mathbf{v}_1$$

and

$$D = Ax_0 + By_0 + Cz_0.$$

Area of parallelograms, part 2

The area of $\mathcal{P}$ is

$$|\mathbf{v}_0 \times \mathbf{v}_1| = \sqrt{A^2 + B^2 + C^2}.$$

If we project this figure into the xy plane, we get a parallelogram with one vertex at $P_0(x_0, y_0, 0)$ and sides given by the vectors

$$\mathbf{w}_0 = \langle a_0, b_0, 0\rangle \quad \mathbf{w}_1 = \langle a_1, b_1, 0\rangle$$

whose area is

$$|\mathbf{w}_0 \times \mathbf{w}_1| = |\langle 0, 0, C\rangle| = |C|$$

The ratio of the area of the projection to the area of $\mathcal{P}$ is

$$\frac{|C|}{\sqrt{A^2 + B^2 + C^2}}.$$

Note that this quantity does not change if A, B, C, and D are all multiplied by a number λ to obtain a different equation of the same plane.

Area of parallelograms, part 3

Also note that the projection has area zero if $C = 0$, which says that the equation of the plane does not depend on z, or that the plane is perpendicular to the xy plane.

On the other hand, if $C \neq 0$, we can invert this ratio to find the amount that the area of the projection should be multiplied by to obtain the area of $\mathcal{P}$. This ratio has the form

$$\sqrt{\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 + 1},$$

so it is always at least 1. The quantities A/C and B/C in this formula are the negatives of the coefficients of x and y when the equation of the plane is solved for z.

Any coordinate plane could be used in place of the xy plane. One consequence of this is a "Pythagorean Theorem for areas": the square of the area of a plane figure is the sum of the squares of its projections into the coordinate planes.

Exercises 12.4

#25. Find vector orthogonal to plane through

$$P(1,0,0,),\quad Q(0,2,0),\quad R(0,0,3)$$

and the area of $\triangle PQR$.

#27.Find vector orthogonal to plane through

$$P(0,0,0),\quad Q(1,-1,1),\quad R(4,3,7)$$

and the area of $\triangle PQR$.

Area on the graph of a function, part 1

The main principle in the differential calculus of functions of several variables is that, if you confine yourself to a set of small enough diameter, any reasonable function is approximately linear. If you really believe this, you are led to the conclusion that the area of of the part of the graph of $f(x,y)$ over a neighborhood of a point P_0 in the xy-plane can be approximated by considering the corresponding area in the tangent plane at the point of this surface above P_0. The sum of such areas for a partition of a region $\mathcal{D}$ in the xy-plane is a Riemann sum of

$$\iint_{\mathcal{D}} \sqrt{f_1(x,y)^2 + f_2(x,y)^2 + 1}\, dA. \tag{A}$$

The quantity dA in this formula stands for $dx\,dy$ or $r\,dr\,d\theta$ in rectangular and polar coordinates, respectively.

Area on the graph of a function, part 2

A rigorous study of surface area is very difficult. Riemann integrals are defined in terms of a very general type of limit. The integrals exist under some fairly general assumptions, guaranteeing that the Riemann sums approximate the value of the integral found by calculus if the partition is fine enough. However, the Riemann sums just constructed involve approximating the surface by pieces that don't fit together to form an approximate surface. If we want to believe that this integral really does give surface area, it would be nice to connect it with the area of something that resembled the surface. A number of reasonable ideas for constructing such measurements turn out to be more general than Riemann sums and often fail to have limits. Although the values obtained from the integral (A) turn out to be correct whenever the integral makes sense, we cannot do a better job of relating them to geometric measurements of the surface.

Area on the graph of a function, part 3

There are only a few examples included in the exercises. The difficulty here is that most of the integrals obtained from (A) cannot be evaluated in terms of familiar functions. This difficulty was already present in connection with arc length. For example, the integral giving the perimeter of an ellipse usually cannot be expressed in terms of familiar functions. The use of (A) to set up an integral representing a surface area is one possible exercise. The warning, "Do not attempt to evaluate the integral", is given in such cases to signify that the result is not likely to be expressible in terms of familiar functions.

Surfaces of revolution, part 1

We show that this formula is consistent with the one used in section 10.3. For a surface of revolution given by a function, we have

$$z = f(r) \quad r^2 = x^2 + y^2.$$

Then $z_x = f'(r)r_x$ and $z_y = f'(r)r_y$ from the first equation, while $r_x = x/r$ and $r_y = y/r$ from the second. Thus, the area of the surface is given by integrating

$$\begin{aligned}\sqrt{1 + z_x^2 + z_y^2} &= \sqrt{1 + \big(f'(r)\big)^2 \left(\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right)} \\ &= \sqrt{1 + \big(f'(r)\big)^2}\end{aligned}$$

with respect to area in xy-plane.

Surfaces of revolution, part 2

There is no special significance to the xy plane in these calculations. It is equally easy to use one of the other coordinate planes and the formulas will involve a different component of the vector perpendicular to the surface. Thus, if $\langle A, B, C\rangle$ is perpendicular to a surface at a point, then the element of surface area at that point dS satisfies

$$\frac{dS}{\sqrt{A^2 + B^2 + C^2}} = \left|\frac{dx\,dy}{C}\right| = \left|\frac{dx\,dz}{B}\right| = \left|\frac{dy\,dz}{A}\right|$$

Exercises 15.6

#3. Find the area of the portion of the plane $3x + 2y + z = 6$ that lies in the first octant.
#9. Find the area of the potion of $z = xy$ that lies inside $x^2 + y^2 = 1$.

Area on a sphere, part 1

The ability to compute surface area allows us to measure familiar objects and refine our intuition about them. The formulas allow us to easily compute the area of the portion of the sphere between two parallels of latitude. Of course, the area of such a region is larger near the equator than near the pole, but a quantitative description will allow it to be compared to other measurements of the figure, and we will find a value for the area that is striking in its simplicity.

Following the conventions of *spherical and cylindrical coordinates*, a sphere of radius ρ will be formed by rotating a circle in an rz-plane (with the positive r-axis horizontal and pointing to the right and the positive z-axis pointing upward) of that radius about the z-axis. Using a convention different from geography, the circle will be parameterized using the angle ϕ measured clockwise from the positive z-axis. This gives

$$z = \rho\cos\phi \text{ and } r = \rho\sin\phi,$$

and the range $0 \le \phi \le \pi$ gives the semicircle in the right half plane.

Area on a sphere, part 2

Rotating this semicircle all the way around the z-axis (through an angle of 2π) gives the whole sphere. The region we want is given by rotating an arc between two fixed values of ϕ all the way around the z-axis. As with any problem involving figures of rotation, the parameterization of the surface will involve whatever parameter is used to describe the curve in the rz-plane that is to be rotated and a second parameter θ (longitude) giving the angle of rotation to reach the actual point on the surface. If the integration with respect to θ is done last, the symmetry of the surface assures us that it will be the integral of a constant from 0 to 2π, so that it will only multiply the value of the inner integral by 2π. This allowed many of these examples to be included in single variable calculus using a *hand-waving* argument to describe the role of rotation.

Area on a sphere, part 3

The parameterization of the surface (in rectangular coordinates) is

$$\langle \rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi\rangle$$

and the partial derivatives with respect to ϕ and θ are

$$\langle \rho\cos\phi\cos\theta,\ \rho\cos\phi\sin\theta,\ -\rho\sin\phi\rangle,$$
$$\langle -\rho\sin\phi\sin\theta,\ \rho\sin\phi\cos\theta,\ 0\rangle.$$

The cross product of these simplifies to

$$\left\langle \rho^2\sin^2\phi\cos\theta,\ \rho^2\sin^2\phi\sin\theta,\ \rho^2\sin\phi\cos\phi\right\rangle.$$

which is a vector of length $\rho^2 \sin\phi$ in the radial direction. This length is the integrand in the surface area integral. Integrating with respect to ϕ gives the difference in values of $\rho^2 \cos\phi$ (since ρ is constant), and integrating with respect to θ multiplies by 2π as has already been noted.

Area on a sphere, part 3

Thus the band between $\phi = \phi_0$ and $\phi = \phi_1$ has area

$$(2\pi\rho)(\rho\cos\phi_1 - \rho\cos\phi_0).$$

The first factor is the length of a great circle. The second factor is $z_1 - z_0$. In particular, if our sphere were surrounded by a cylinder of equal radius and our region projected to that cylinder from the common axis of the two figures, the spherical band and its projection on the cylinder would have equal area.

Lecture #14

Parametric surfaces, part 1

Section 16.6 extends the study of surface area to parametric surfaces.

Just as curves in space are best described by giving a vector function $\mathbf{r}(t)$ that may be thought of as describing how the curve is drawn, so surfaces should be given by a function $\mathbf{r}(u, v)$ expressing the space coordinates x, y, and z in terms of two parameters u and v that play the role of coordinates on the surface.

Surfaces of revolution are naturally parameterized by the adding an angular parameter θ to the parameter that draws the curve being rotated. If you are rotating about one of the coordinate axes, the distance along that axis is part of the parametric description of the curve, and the other two coordinates are obtained by multiplying the distance to that axis (the other part of the parametric description) by $\cos\theta$ and $\sin\theta$.

Parametric surfaces, part 2

You can also get the surface traced out by the tangent lines to a space curve by letting $\mathbf{r}(u)$ be the curve and v the parameter that draws the tangent line at $r(u)$. You can also get the surface traced out by the tangent lines to a space curve by letting $\mathbf{r}(u)$ be the curve and v the parameter that draws the tangent line at $r(u)$.

Exercises 16.6

#39 Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ the lies inside the cylinder $x^2 + y^2 = ax$.

Integrals of vector fields over surfaces, part 1

Many physical applications involve vector fields interpreted as **flows** and require the measurement of the **flux** through a surface. The contribution to the flux of a small piece of the surface should be the product of its area and the component of the flow perpendicular to it (since flow parallel to the surface does not cross it). There is an underlying assumption that the flow is a **vector** quantity and that it behaves in a **linear** fashion when vectors are added or multiplied by scalars. For anything with this behavior, it is essential that it be measured by a quantity that is sensitive only to normal components. These integrals are also **oriented**, in the sense that reversing the direction of the flow should give the negative of the previous measurement. This required that the surfaces have a clear **inside** and **outside**. In many of the exercises, the surface is the graph of a function $z = f(x, y)$ and an **upward** direction (i.e., a positive third component) can be used as a substitute for "outward".

Integrals of vector fields over surfaces, part 2

There are surfaces that do not have a global orientation. That it, you can walk around the surface carrying a continuously varying unit normal vector and get back to the same point with the normal having reversed its direction. The Möbius band is a common example: a rectangle is given a half-twist before gluing its ends together, so points that were on one side of the surface now find themselves next to points that were originally on the other side. We won't do anything with such surfaces except to acknowledge their existence.

In a later section, we will use surface integrals around closed surfaces to measure properties of the region inside the surface. For such results to make any sense, the surface must *have* an inside — i.e., it must have a global orientation. This is an extra concern in formulating theorems in this area, but it turns out not to cause any real trouble.

Computing surface integrals, part 1

When you want to do calculus, you determine what the expression you are computing looks like in the special case of the graph of a function. Since we are now concerned with surfaces, this means $z = f(x, y)$. We have already noted that a consistent orientation can be provided by the notion of **upward**. It remains only to convert the flux integral

$$\iint \mathbf{F} \cdot \mathbf{n} \, dS$$

into something that we can compute.

Although the notation suggests that we should find $\mathbf{n}$ and dS separately and substitute our findings into the definition, these two terms should be considered as a single object. The reason for this is the formula from the last lecture

$$\frac{dS}{\sqrt{A^2 + B^2 + C^2}} = \frac{dx\,dy}{C} = \frac{dx\,dz}{B} = \frac{dy\,dz}{A}$$

Computing surface integrals, part 2

The homogeneous nature of this formula means that $\langle A, B, C\rangle$ can be taken to be any normal vector.

This isn't quite right since we are integrating a quantity that may change sign and we need some way to be sure that our answer has the correct sign. This is best dealt with be making the process completely formal, but it is difficult to appreciate the abstraction required until you have some experience where it is required.

If you multiply by $\langle A, B, C\rangle$, the first expression is exactly $\mathbf{n}\,dS$, and the others are the expressions to be used in computation. In one version, $\mathbf{F}$ is written as $\langle P, Q, R\rangle$ and the separate terms are written in the simplest form to get

$$\iint P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy. \qquad (*)$$

Computing surface integrals, part 3

Another form which is suitable for graphs of functions used the vector with $C = 1$ as a normal to get

$$\iint \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dx\,dy.$$

The connection between these expressions is given by replacing dz by its expression in terms of dx and dy and treating $dx\,dx$ or $dy\,dy$ as zero, and $dy\,dx$ as $-dx\,dy$.

Exercises

Find $\iint \mathbf{F}\cdot\mathbf{n}\,dS$ over the given region (with upward orientation).

#19. $\mathbf{F} = \langle xy, yz, zx\rangle$ on $z = 4 - x^2 - y^2$ above $0 \le x \le 1$, $0 \le y \le 1$.

#21. $\mathbf{F} = \langle xye^y, -xze^y, z\rangle$ on the portion of $x + y + z = 1$ in the first octant.

An example using $(*)$ is

#27. $\mathbf{F} = \langle x, 2y, 3z\rangle$ on the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ with outward orientation.

Another problem using unusual roles for the coordinate variables is

#25. $\mathbf{F} = \langle 0, y, -z\rangle$ with the outward orientation on the closed surface formed by the paraboloid $y = x^2 + z^2$ for $0 \le y \le 1$ and the disk $x^2 + z^2 \le 1$ in the plane $y = 1$.

Lecture #15

The setting for Stokes' Theorem

It is conventional to state theorems before proving them, but this sometimes leads to unmotivated work establishing the definitions needed to state the result. Since **the motivation lies in the proof**, we give the proof first, and then interpret it. We use (x, y, z) for the coordinates in the $\mathbb{R}^3$ where all objects are constructed.

One part of the theorem concerns a **surface** $\mathcal{S}$ that we begin by assuming to be **the graph of a function** $z = g(x, y)$. This suffices for our needs since we need only have enough of a proof to provide clues to the correct statement of the theorem and interpretations of the formulas it relates.

Another part of the theorem requires a **closed curve** $\mathcal{C}$ **lying in the surface** $\mathcal{S}$. As usual, $\mathcal{C}$ is assumed to be **given by a parameterization**

$$\mathbf{r}(t) = \big\langle x(t), y(t), z(t) \big\rangle$$

Since $\mathcal{C}$ **is contained in** $\mathcal{S}$, we must have $z(t) = g\big(\, x(t), y(t) \,\big)$. Also, let $\mathcal{C}_0$ be the **projection** of $\mathcal{C}$ into the xy-plane, so that it is parameterized by $\langle x(t), y(t) \rangle$.

The line integral in Stokes' Theorem

One side of the Stokes Theorem equation is the **integral of a vector field F** around $\mathcal{C}$.

Write the **vector field** as

$$\mathbf{F} = \big\langle P(x, y, z),\ Q(x, y, z),\ R(x, y, z) \big\rangle.$$

Then $\displaystyle\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is an abbreviation for the expression

$$\begin{aligned}\int_a^b & P\big(\, x(t), y(t), z(t) \,\big)\, x'(t) + \\ & Q\big(\, x(t), y(t), z(t) \,\big)\, y'(t) + \\ & R\big(\, x(t), y(t), z(t) \,\big)\, z'(t) \quad dt,\end{aligned}$$

where the values $t = a$ and $t = b$ correspond to **going once around** $\mathcal{C}$. This integral is a sum of three terms that can be treated separately.

Projection

We next write expressions for the interpretations of these integrals as **line integrals on** $\mathcal{C}_0$. This involves replacing every mention of z by $g(x, y)$. **This is straightforward** in the first two terms, but in the third term,

$$z'(t) = g_1\big(\, x(t), y(t) \,\big)\, x'(t) + g_2\big(\, x(t), y(t) \,\big)\, y'(t).$$

where g_i indicates the **partial derivative of the function** g with respect to its i^{th} argument. Omitting explicit mention of t, the integral can be written as a **line integral**

$$\begin{aligned}\oint_{\mathcal{C}_0} & \Big(P\big(\, x, y, g(x, y) \,\big) + R\big(\, x, y, g(x, y) \,\big)\, g_1(x, y)\Big)\, dx + \\ & \Big(Q\big(\, x, y, g(x, y) \,\big) + R\big(\, x, y, g(x, y) \,\big)\, g_2(x, y)\Big)\, dy.\end{aligned}$$

This line integral in the xy plane is equal, by **Green's Theorem**, to the **double integral over the region** $\mathcal{S}_0$ bounded by $\mathcal{C}_0$ of

$$\frac{\partial}{\partial x}\big(Q\big(x,y,g(x,y)\big)+R\big(x,y,g(x,y)\big)g_2(x,y)\big)-$$
$$\frac{\partial}{\partial y}\big(P\big(x,y,g(x,y)\big)+R\big(x,y,g(x,y)\big)g_1(x,y)\big).$$

Simplifying the integrand

$$\begin{aligned}
&\frac{\partial}{\partial x}\big(Q\big(x,y,g(x,y)\big)+R\big(x,y,g(x,y)\big)g_2(x,y)\big)=\\
&\quad Q_1\big(x,y,g(x,y)\big)+Q_3\big(x,y,g(x,y)\big)g_1(x,y)+\\
&\quad R\big(x,y,g(x,y)\big)g_{21}(x,y)+R_1\big(x,y,g(x,y)\big)g_2(x,y)+\\
&\quad R_3\big(x,y,g(x,y)\big)g_1(x,y)g_2(x,y)\\
&\quad \text{and}\\
&\frac{\partial}{\partial y}\big(P\big(x,y,g(x,y)\big)+R\big(x,y,g(x,y)\big)g_1(x,y)\big)=\\
&\quad P_2\big(x,y,g(x,y)\big)+P_3\big(x,y,g(x,y)\big)g_2(x,y)+\\
&\quad R\big(x,y,g(x,y)\big)g_{12}(x,y)+R_2\big(x,y,g(x,y)\big)g_1(x,y)+\\
&\quad R_3\big(x,y,g(x,y)\big)g_1(x,y)g_2(x,y)
\end{aligned}$$

Interpreting the planar integral

The terms containing second derivatives of g or products of two derivatives of g in this expression cancel, and the remaining terms may be grouped as

$$(R_2-Q_3)(-g_1)+(P_3-R_1)(-g_2)+(Q_1-P_2).$$

The second factors in these terms are the components of $\langle -g_1,-g_2,1\rangle$ which is perpendicular to $\mathcal{S}$. The first factor must then be the components of a vector field being integrated over the surface. Note that the normalization of our normal vector to have third coordinate 1 gives the expression for a flux integral for the upward orientation with respect to $dx\,dy$.

The other factor is a vector field constructed from derivatives of **F**. This expression is often described as $\nabla\times\mathbf{F}$ since the pattern of partial derivatives follows the same pattern as terms in a cross product. This construction is called the **curl** of **F**.

From particular to general

The previous analysis applies on any piece of the surface where z **can be given** as a function of x and y. A computation of the surface integral using parameters other than x and y only requires a different **scaling** of the normal vector.

For example, the rightward orientation for an integral with respect to y and z would have first coordinate $+1$.

All of our oriented integrals are such that cutting the region into pieces gives the integral as the sum over the pieces. The implicit function theorem tells us that a nonzero component of the normal at a point allows a small piece containing the point to be found on which the selected variable is a function of the remaining variables. The whole surface is thus broken into pieces covered by our special case.

Conservative vector fields

One observation connected to Stokes' Theorem is that the components of $\nabla \times \mathbf{F}$ are exactly the things that must be tested to show that $\mathbf{F}$ is **conservative**. Since the line integral of a conservative field is zero around every curve, it is natural to expect that its corresponding surface integral would also be identically zero. As in the case of Green's Theorem, one can use Stokes' Theorem to replace a line integral by a surface integral to remove conservative vector fields that may complicate the computation without affecting the value of the integral.

Although we will see that surface integrals usually depend on the surface, the surface integrals in Stokes' Theorem give the same value for integrals over any surfaces having the same boundary curve.

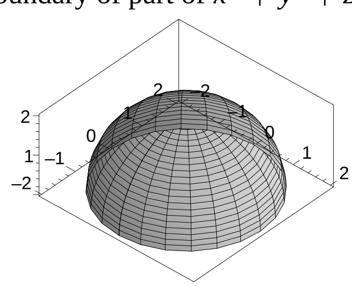
Conventions for exercises

In the exercises for section 16.8, either a curve or a surface is given, and the orientation is such that the surface has an **upward** normal (i.e, the **third component of the normal is positive**). Equivalently, the curve is traced in the counterclockwise direction when viewed from above. In all cases, the vector field $\mathbf{F}$ that appears in the line integral is given.

We ignore the instructions and examine various ways to compute the integral.

Exercise 16.8#3

$\mathbf{F} = \langle x^2 e^{yz}, y^2 e^{xz}, z^2 e^{xy} \rangle$, $\mathcal{C}$ is boundary of part of $x^2 + y^2 + z^2 = 4$ where $z \geq 0$.



Solution 16.8#3, part 1

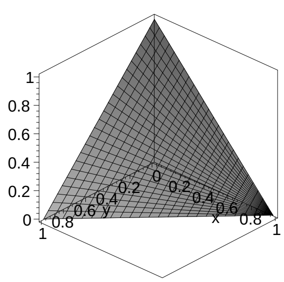
$\mathcal{C}$: $\mathbf{r} = \langle 2\cos t, 2\sin t, 0\rangle$ for $0 \le t \le 2\pi$
$d\mathbf{r} = \langle -2\sin t, 2\cos t, 0\rangle dt$
$\mathbf{F}$ on $\mathcal{C}$: $\langle 4\cos^2 t, 4\sin^2 t, 0\rangle$
$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int -8\cos^2 t\sin t + 8\sin^2 t\cos t\,dt = 0$
$\nabla\times\mathbf{F} = \langle xz^2e^{xy} - xy^2e^{xz}, x^2ye^{yz} - yz^2e^{xy}, y^2ze^{xz} - x^2ze^{yz}\rangle$
To integrate this over the sphere, we would probably choose to use a parametric description using longitude and latitude. The normal vector is in the direction of $\langle x, y, z\rangle$

Solution 16.8#3, part 2

But there is a **better way**. Stokes' Theorem allows any surface with boundary $\mathcal{C}$ to be used, so we can take $\mathcal{S}$ to be a circular disk in the xy plane. This simplifies the calculation in two ways: **first**, it substitutes $z = 0$ into $\nabla\times\mathbf{F}$; **second**, $\mathbf{n}\,dS = \langle 0, 0, 1\rangle dx\,dy$. Together, this makes the integrand zero, so the integral is zero.

Exercise 16.8#7

$\mathbf{F} = \langle x + y^2, y + z^2, z + x^2\rangle$ $\mathcal{C}$ is triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$.



Solution 16.8#7

$\mathcal{C}$: the three segments $\mathbf{r} = \langle 1-t, t, 0\rangle$, $\langle 0, 1-t, t\rangle$, $\langle t, 0, 1-t\rangle$ each with $0 \le t \le 1$. On each of these, $d\mathbf{r}$ is a constant vector times dt — a different constant for each segment.

$\mathbf{F}$ on $\mathcal{C}$: a quadratic function of t on each segment. We don't go any farther until exploring the surface integral.

$\nabla\times\mathbf{F} = \langle -2z, -2x, -2y\rangle$ and $\mathbf{n}\,dS = \langle 1, 1, 1\rangle dx\,dy$.

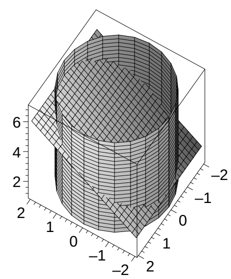
Thus, $\nabla\times\mathbf{F}\cdot\mathbf{n}\,dS$ is $-2(x + y + z)\,dx\,dy$ on $x + y + z = 1$. Thus, its integral over $\mathcal{S}$ is the integral of -2 over the projection of the triangle in the xy plane. The projection has area $1/2$, so the integral is -1.

To check, the integral on each side if the triangle is

$$\int_0^1 -1 + 2t - t^2\,dt = {}^{-1}\!/_3.$$

Exercise 16.8#9

$\mathbf{F} = \langle 2z, 4x, 5y \rangle$, $\mathcal{C}$ is intersection of $z = x + 4$ with $x^2 + y^2 = 4$.



Solution 16.8#9, part 1

$\mathcal{C}$: $\mathbf{r} = \big\langle 2\cos t, 2\sin t, 2\cos t + 4 \big\rangle$ for $0 \le t \le 2\pi$

$d\mathbf{r} = \big\langle -2\sin t, 2\cos t, -2\sin t \big\rangle dt$

$\mathbf{F}$ on $\mathcal{C}$: $\big\langle 4\cos t + 8, 8\cos t, 10\sin t \big\rangle$. The integral involves only low powers of trigonometric functions, so it would not be difficult to evaluate directly, but we first consider the surface integral given by Stokes' theorem.

$\nabla \times \mathbf{F} = \big\langle 5, 2, 4 \big\rangle$.

Since $\mathcal{S}$ can be taken to lie in the plane $z = x + 4$, $\mathbf{n}\,dS$ can be taken to be $\big\langle -1, 0, 1 \big\rangle\,dx\,dy$ and $\nabla \times \mathbf{F} \cdot \mathbf{n}\,dS = -1\,dx\,dy$.

The surface integral is the negative of the area of the projection into the xy plane. That projection is defined by the cylinder $x^2 + y^2 = 4$, so it is a circle of radius 2. The integral is -4π.

Solution 16.8#9, part 2

Direct evaluation of the line integral leads to

$$\int_0^{2\pi} -8\sin t\cos t - 16\sin t + 16\cos^2 t - 20\sin^2 t\,dt.$$

This agrees with the value of the surface integral.

Lecture #16

Parametric surfaces

We continue to assume that a surface is given by $x = g(x, y)$. Then, both the **surface area** and **flux** integrals can be expressed as integrals over the **projection** in the xy plane.

If the surface is **really** given by a **parametrization** with x, y and z expressed in terms u and v, then, the **change of variables** formula allows the integrals in terms of x and y to be written in terms of u and v. The key formula is

$$dx\,dy = \left(\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}\right) du\,dv$$

The parametric form of the integrals, part 1

In flux integrals, the quantity $\mathbf{n}\,dS$ becomes

$$\big\langle -g_1, -g_2, 1\big\rangle\,dx\,dy$$

on the graph of a function. To see what this becomes for parametric surfaces, one should multiply each component of this vector by

$$\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

The combination $g_1 \partial x/\partial u$ appearing in the first component **suggests** $\partial z/\partial u$, but the **chain rule** says

$$\frac{\partial z}{\partial u} = g_1\frac{\partial x}{\partial u} + g_2\frac{\partial y}{\partial u}$$

The parametric form of the integrals, part 2

Although there is an extra term in this expression, when **both** terms are considered, we have

$$\frac{\partial y}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial u} = -g_1\left(\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}\right)$$

The second component behaves similarly. If the surface is given parametrically, the resulting expressions depend only on the parametric description and not on the (possibly fictitious) function g.

The resulting formula is

$$\mathbf{n}\,dS = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,du\,dv.$$

Flux integrals

The **change of variables** formula allows $\mathbf{n}\,dS$ to be written

$$\langle dy\,dz, dz\,dx, dx\,dy \rangle$$

where each is to be expressed in terms of the parameters u and v. Because of this interpretation, **orientation must be considered**. Replacing $dx\,dy$ by $dy\,dx$ changes the sign.

For area on a sphere, as in lecture 13, one gets the outward orientation using $d\phi\,d\theta$.