

Mathematics 251 Assorted exercises
Lines, areas, double integrals and applications
Spring 2005

Lines

1. An arbitrary direction in the xy -plane is given by a vector $\mathbf{v} = \langle a, b \rangle$. Use the **cross product** $\mathbf{k} \times \mathbf{v}$ to produce a vector in the xy -plane perpendicular to \mathbf{v} . If (x_0, y_0) is a point on the line, then **parametric** equations of the line are given by $\langle x, y \rangle = \langle x_0, y_0 \rangle + t\mathbf{v}$, but the line also has an equation of the form $Ax + By = C$. Here, $\langle A, B \rangle$ is a vector perpendicular to the direction of the line. You already have such a vector. For the vector you have, determine C in terms of a, b, x_0, y_0 .

2. Using either the formula obtained in exercise 1 or the method used to obtain it, find an equation of the form $Ax + By = C$ for the line through $P: (1, 2)$ and $Q: (6, -1)$. Use \overrightarrow{PQ} as a vector in the direction of the line.

Areas

3. The equation above was obtained by equating

$$\langle A, B \rangle \cdot \langle x - x_0, y - y_0 \rangle$$

to zero. This quantity is $\sqrt{A^2 + B^2}$ times the distance to the line, but $\sqrt{A^2 + B^2}$ is the length of the interval between the points P and Q determining the line. Hence, the value of $Ax + By - C$ at a point R is twice the area of $\triangle PQR$ (with a sign that indicates the **orientation** of the triangle). Take $R: (3, 1)$ with P and Q given in exercise 2, and find the area of $\triangle PQR$. Compare this to the general formula using $\overrightarrow{PQ} \times \overrightarrow{PR}$.

4. Green's theorem shows that

$$\frac{1}{2} \oint_{\mathcal{C}} x dy - y dx$$

gives the area of the region bounded by \mathcal{C} . Apply this to the triangle with vertices $(0, 0)$, (x_0, y_0) and (x_1, y_1) . First show that the integral is zero along any segment of a line through the origin. This shows that the area is given by the integral along the segment joining (x_0, y_0) and (x_1, y_1) . Find this integral: the result should look familiar.

Moments

5. Determine

$$\iint_{\mathcal{T}} y \, dA$$

where \mathcal{T} is the triangle with vertices $(0, 0)$, (a_0, b) and (a_1, b) by integrating first with respect to x . As noted in the **Theorems about moments** section of the lecture of Feb. 21, this gives the area of the triangle times the y coordinate of the centroid, and the **geometric interpretation** of this calculation identifies the location of the centroid. Your result should be consistent with the centroid being at the point that is $\frac{1}{3}$ of the **vector sum** of the vertices of the triangle.

6. This tells us that, for \mathcal{B} being the triangle with vertices $(0, 0)$, $(6, 0)$ and $(0, 3)$, the area is 9 and the centroid is at $(2, 1)$, so that

$$\iint_{\mathcal{B}} dA = 9, \quad \iint_{\mathcal{B}} x \, dA = 54, \quad \iint_{\mathcal{B}} y \, dA = 27.$$

In other words, the integrals with respect to area of every polynomial of degree at most 1 over \mathcal{B} is known. Then, Green's theorem says that

$$\oint_{\mathcal{C}} (5x^2 - 2xy + 3x) \, dx + (2x^2 + xy - 4x) \, dy, \quad (*)$$

where \mathcal{C} is the boundary of \mathcal{B} (in the positive sense), can be expressed in terms of the integrals that we have tabulated. Thus, the line integral can be found using **no further integration**. Find the value of the line integral (*).

7. By evaluating suitable double integrals, find the area and location of the centroid of the region above $y = x^2$ and below $y = 1$. (A simple change of variables will allow this special case to be applied to an arbitrary region between a parabola and a line.)