Arc length

If the vector function $\mathbf{r}(t)$ is thought of as giving position as a function of time, then its derivative $\mathbf{r}'(t)$ gives **velocity**. The length of $\mathbf{r}'(t)$ measures the **speed** and we shall see that the distance traveled along the curve is the integral of speed. We have already met $\mathbf{T}$, which is a unit vector in the direction of $\mathbf{r}'(t)$. If a different parameter u is used to describe the curve, with t being an increasing function of u, $d\mathbf{r}/du = (dt/du)(d\mathbf{r}/dt)$. The first factor is a scalar, so it does not affect $\mathbf{T}$.

The usual approach to measuring the length of a curve $\mathbf{r}(t)$ between the point where $t = a$ and the point where $t = b$ is to select values $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ and find the length of the polygonal path connecting the points $\mathbf{r}(t_i)$ in order. This gives a sum of terms of the form

$$\Delta t\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2+\left(\frac{\Delta y}{\Delta t}\right)^2+\left(\frac{\Delta z}{\Delta t}\right)^2}.$$

With any kind of luck, this will approach

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

as the t_i get closer together. We do not prove this formula, but we accept it as sufficiently plausible to be taken as a definition of arc length.

One encouraging fact is that, when the parameterization is changed by the substitution $t = g(u)$ with a monotonic function g, the integral doesn't change. That is, our way of finding the length of the curve depends on the curve and not on how it is drawn.

Exercises

There are only a few special curves for which this integral can be evaluated in closed form. Note how the examples here simplify.

$$x = 2\sin t \quad y = 5t \quad z = s\cos t \tag{1}$$

$$x = \sqrt{2}t \quad y = e^{t} \quad z = e^{-t} \tag{3}$$

Parameterization by arc length

As long as we know that a function is defined, whether or not we have previously named it, it is available for use. The arc length integral always gives arc length s as a function of the original parameter t. The derivative ds/dt is the integrand of the arc length integral, which is always positive. Thus, s is an increasing function of t, and there is an inverse function giving t in terms of s. This parameterization is often used to give a geometric definition of a quantity that we intend to study.

In order to compute such quantities, a substitution is made to express it in terms of the parameter t appearing in the original definition of the curve. In particular, $\mathbf{T}$ is the derivative of the position vector with respect to arc length.

A useful result

The usual rules of calculus for sums and products are easily proved for derivatives of vectors. One consequence of this is that if $\mathbf{a}(t)$ is of constant length, so that $\mathbf{a}(t) \cdot \mathbf{a}(t)$ is a constant function, then

$$0 = \mathbf{a}(t) \cdot \mathbf{a}'(t) + \mathbf{a}'(t) \cdot \mathbf{a}(t) = 2\mathbf{a}(t) \cdot \mathbf{a}'(t)$$

so that $\mathbf{a}'(t)$ is always perpendicular to $\mathbf{a}(t)$.

Curvature

From the last result, we get that $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$. The direction of $\mathbf{T}'(t)$ is called the **principal normal** and denoted $\mathbf{N}$. Changing the parameter multiplies the derivative of $\mathbf{T}$ by a scalar (positive if the parameters are increasing functions of one another), so $\mathbf{N}$ is independent of the parameterization. If you take **arc length** as the parameter, then the magnitude of the derivative is also significant. This value is called **curvature**, and denoted κ, here described by **definition (8)**. Finally, it is not actually necessary to construct this parameterization, since the value at any point can be found from the chain rule. This gives **formula (9)**, which we use in the example below. However, this gives all geometric features as functions of the original parameter.

The main example

If $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, describing a circle of radius a in the xy plane, $\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$, and we can **see** its length and direction: $ds/dt = a$ and $\mathbf{T} = \langle -\sin t, \cos t\rangle$. Then

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{\langle -\cos t, -\sin t\rangle}{a}$$

in this case. Geometrically, we see that $\mathbf{N}$ is a unit vector pointing towards the center of the circle, and $\kappa = 1/a$.

Exercises

Some exercises for finding **T**, **N** and κ are

$$x = {}^1\!/\!{}_3 t^3 \quad y = t^2 \quad z = 2t \tag{13}$$

$$x = \sin t \quad y = \cos t \quad z = \sin t \tag{17}$$

Motion in Space

If $\mathbf{r}(t)$ represents the **position** of a body as a function of time, then $\mathbf{r}'(t)$ is **velocity** and $\mathbf{r}''(t)$ is **acceleration**. The key idea in Newton's explanation of motion was that motion represented the effect of **forces** and **force** is **mass** times **acceleration**. All of these quantities except **mass** are vectors; **mass** is a scalar that is constant for ordinary objects. If you observe the position function $\mathbf{r}(t)$, you determine the acceleration $\mathbf{r}''(t)$ and use that to help identify the force.

Components of acceleration

The formula

$$\mathbf{r}''(t) = \frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'.$$

was derived in Section 13.3 (page 865). We also have $\mathbf{T}' = \kappa(ds/dt)\mathbf{N}$, so

$$\mathbf{r}''(t) = \frac{d^2 s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$

Writing v in place of ds/dt gives formula (7) of Section 13.4. The quantity v represent the **speed** of the object. The vectors **T** and **N** are perpendicular unit vectors that are part of a coordinate system that moves with the object. In particular

T is "straight ahead". In this coordinate system, the first part of the expression for $\mathbf{r}''(t)$ describes the part of the acceleration (and, hence, of the force) that leads to a change of speed, while the second part describes the part of the acceleration that leads to a change of direction. These two terms are important in the way that motion is perceived, so it is important to see how they can be computed. Since there are many different approaches to finding the quantities in this formula, it is useful to point out that calculations done earlier with numerical vectors give a way to organize the work efficiently. One needs only connect the use of the word **component** here with the earlier use of that word.

The velocity vector $\mathbf{v}(t) = v(t)\mathbf{T}(t)$, so it defines the direction $\mathbf{T}(t)$. In the previous sense of the word, the component of $\mathbf{a}(t)$ in the direction of $\mathbf{v}(t)$ is what we call here “the tangential component of acceleration”, so it will be equal to d^2s/dt^2 even if its computation does not appear to involve differentiation of $v(t) = ds/dt$. It you have this component, you also have the projection by multiplying by the vector $\mathbf{T}(t)$. From the whole vector and the tangential projection, you can find the projection on the principal normal and its component. This leads to an **algorithm** for computing curvature that is not easily summarized in a formula, but may be simpler than the formulas of section 13.3.

Exercises

The instructions are to find the tangential and normal components of acceleration.

$$\mathbf{r}(t) = \left\langle 3t - t^3, 3t^2 \right\rangle \tag{29}$$

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle \tag{31}$$