Functions of several variables

Expressions like $x + y$ and xy have been used throughout our study of calculus.

We now admit that these expressions are functions of the two variables x and y. The familiar rules for differentiating sums and products will be seen to be special cases of a chain rule.

Notation

We use names like $f(x, y)$ to designate expressions depending on the two variables x and y, such as $x + y$, xy, or

$$y(1 - 8xy)e^{-2x^2 - y^2}$$

In such expressions, the pair (x, y) can be thought of as representing a point in the plane and the graph of $z = f(x, y)$ will define a surface in (x, y, z) space.

As with functions of a single variable, the **domain** of f is the set of points (x, y) at which we are allowed to evaluate the function.

Functions and expressions

It is a common **abuse of notation** to identify a function f with the resulting expression $f(x, y)$. This is not good practice. A function of two variables f may be defined by giving $f(x, y)$, say

$$y(1-8xy)e^{-2x^2-y^2}$$

and this can be used to find $f(u, v)$ or even $f(y, x)$. Maple knows the difference and has different ways of working with **functions** and **expressions**.

A key example

If

$$g(x, y) = x + y^2,$$

then

$$g(y, x) = y + x^2.$$

One should **never assume** that the variables used to **describe** a function have **any significance** whatsoever. Unfortunately, most of the notations used in the textbook violate this rule.

Sensible Notation

The only notation that doesn’t is one that uses f_1 to stand for the derivative of the function f with respect to its **first variable**. This has the advantage that one can write $f_1(a, b)$ for the result of evaluating this function at the point (a, b), i.e., the result of first differentiating the function and then evaluation the result at the base point.

Higher derivatives

Once one has a derivative, either as an expression or as a function, one can think of differentiating **that**. If one expects to do a lot of that sort of thing, an abbreviation is needed. Thus one writes

$$\frac{\partial^2 z}{\partial x^2} \text{ for } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right); \; \frac{\partial^2 z}{\partial y \partial x} \text{ for } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right);$$

and f_{12} for $(f_1)_2$.

Fortunately, you only need to be careful about the order of the variables in this expression when making the definition, since $f_{12} = f_{21}$ for functions that you will meet in practice.

Exercises 14.3

Find partial derivatives.

$$3x - 2y^4 \tag{11}$$

$$\frac{x - y}{x + y} \tag{15}$$

$$\sqrt{x^2 + y^2} \tag{33}$$

The chain rule

There are other derivatives involving functions of several variables that can be found. Suppose that z is given in terms of x and y, and that x and y are each given in terms of t. You could (and Maple does) use this to find z in terms of t and then calculate $D_t z$ (this is the neatest notation for this discussion). Alternatively, you could apply the rules of differentiation to the expressions that you are given. Whatever the expression for z tells you is the last step in its computation is the first differentiation formula to be applied. In this process, expressions equal to $D_t z$ are obtained that can contain x, y, t, $D_t x$ and $D_t y$. Since x and y are given in terms of t, their expressions are used when you need to expand $D_t x$ and $D_t y$ in terms of t.

An example

$$z = y(1 - 8xy)e^{-2x^2-y^2}$$

and assume that x and y are functions of t to be specified later. Then

$$\begin{aligned} D_t z &= y(1-8xy)D_t e^{-2x^2-y^2} + e^{-2x^2-y^2} D_t(y - 8xy^2) \\ &= y(1-8xy)e^{-2x^2-y^2} D_t(-2x^2 - y^2) \\ &\qquad + e^{-2x^2-y^2}(D_t y - 16xy D_t y - 8y^2 D_t x) \\ &= y(1-8xy)e^{-2x^2-y^2}(-4x D_t x - 2y D_t y) \\ &\qquad + e^{-2x^2-y^2}(D_t y - 16xy D_t y - 8y^2 D_t x) \end{aligned}$$

Linearity

When the result of the example is simplified, each term will have exactly one factor of $D_t x$ or $D_t y$.

Let's take a close look at the formulas of elementary calculus:

$$D_t(x+y) = D_t x + D_t y \tag{S}$$

$$D_t(x \cdot y) = D_t x \cdot y + x \cdot D_t y \tag{P}$$

$$D_t\big(f(x)\big) = f'(x) \cdot D_t x \tag{C}$$

Summary of Differentiation

These rules suffice to differentiate all the functions met so far, when supplemented by special formulas for differentiating functions given by $f(x) =$ one the following expressions: a constant, x^n, e^x, $\ln x$, $\sin x$, or $\cos x$, $\arctan x$. A few more formulas are obtained to avoid deriving them from other formulas every time they are needed, but this short list of formulas is a good summary of elementary calculus.

The Chain Rule

The thing to notice about formulas (S), (P) and (C) is that each **term** contains a **factor** that is a single application of D_t. This means that, in the setting at the start of this lecture,

$$D_t z = A \cdot D_t x + B \cdot D_t y, \tag{$*$}$$

where A and B are expressions involving x and y.

The Chain Rule, page 2

Formula $(*)$ holds independent of the dependence of x and y on t. The special cases used to define partial derivatives (or, at least, their values at particular points) are obtained by using the parameterizations: (1) $x = t$, $y = b$; or (2) $x = a$, $y = t$. This shows that $A = D_x z$ and $B = D_y z$. The usual statements of the chain rule are obtained by translating this statement into different notations.

Proving the chain rule

For functions given by expressions that we recognize, our calculation is a proof that the derivatives exist. In fact, a close examination of what we have when we have reached the stage of formula $(*)$ shows that the surface defined by the given expression of z in terms of x and y has a tangent plane wherever the calculation is valid.

Proving the Chain Rule, page 2

What this approach does **not** do is tell how to deal with a new function of several variables. What should be done, if such a function is ever met, is to show from the definition of the function that it has a "good linear approximation", which leads to tangent plane to the graph of $z = f(x, y)$. This tangent plane can be use to verify the chain rule formula when our function is composed with $x = g(t)$, $y = h(t)$.

Exercises

Find derivatives and identify role of the chain rule.

1. $z = x^2 y + xy^2$, $x = 2 + t^4$, $y = 1 - t^3$.

3. $z = \sin x \cos y$, $x = \pi t$, $y = \sqrt{t}$.

7. $z = x^2 + xy + y^2$, $x = s + t$, $y = st$.

9. $z = \arctan(2x + y)$, $x = s^2 t$, $y = s \ln t$.

19. $w = x^2 + y^2 + z^2$, $x = st$, $y = s \cos t$, $z = s \sin t$ (at $s = 1$, $t = 0$).

Gradients

There is one more thing to be seen in $(*)$. Whenever one has a sum of terms, each of which is a product of something of one type and something of another, it should be viewed as a dot product of vectors. We have already met the vector $\langle D_t x, D_t y\rangle$ as the velocity vector when $\mathbf{r}(t) = \langle x, y\rangle$ gives the position of a point at time t. We collect the other factors into a vector $\left\langle D_x z, D_y z\right\rangle$. When differentiating a function f instead of an expression z, this has the form $\langle f_1, f_2\rangle$. In this form, it is easy to imagine the generalization to functions of any number of variables. This vector is called the **gradient** of f and denoted ∇f.

Gradients, page 2

Gradients are very much a “function thing” since it emphasizes the domain of the function rather than the range — there is no good notation for the same object constructed from an expression.

Like all other derivatives, gradients will be evaluated at points of their domain when they appear in applications.