Gradients

If z is a function of x and Y, which are functions of t, we have seen that there is a chain rule that takes the form

$$D_t z = A \cdot D_t x + B \cdot D_t y, \qquad (*)$$

where A and B are expressions involving x and y.

Gradients, page 2

There is one more thing to be seen in the formula $(*)$. Whenever one has a sum of terms, each of which is a product of something of one type and something of another, it should be viewed as a dot product of vectors. We have already met the vector $\langle D_t x, D_t y\rangle$ as the velocity vector when $\mathbf{r}(t) = \langle x, y\rangle$ gives the position of a point at time t. We collect the other factors into a vector $\left\langle D_x z, D_y z\right\rangle$. When differentiating a function f instead of an expression z, this has the form $\langle f_1, f_2\rangle$. In this form, it is easy to imagine the generalization to functions of any number of variables. This vector is called the **gradient** of f and denoted ∇f.

Gradients, page 3

The ∇f notation for gradients are very much a “function thing” since it emphasizes the domain of the function rather than the range — there is no comparable notation for the same object constructed from the expression for $f(x, y)$.

Like all other derivatives, gradients will be evaluated at points of their domain when they appear in applications.

The standard differentiation formulas may be extended to formulas for gradients of sums, products and compositions.

Planes

Any equation of the form

$$ax + by + cz + d = 0, \tag{P1}$$

where a, b, and c are **not all zero**, describes a plane. If (x_0, y_0, z_0) is any point that satisfies this equation, then $(P1)$ becomes

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{P2}$$

which asserts that $\langle a, b, c \rangle$ is perpendicular to the general line

$$\langle x - x_0, y - y_0, z - z_0 \rangle$$

connecting the base point to another point satisfying the equation.

Planes, part 2

A geometric object is given by an equation when the equation is satisfied by the points belonging to the object and **only** by those points. The simple nature of the equation of a plane is a consequence of the characterization of the directions in a plane as those perpendicular to a given vector. In some cases, the geometric description of a plane will identify that vector, but sometimes it will be necessary to do some computation to find the vector.

The cross product (part 1)

It is more common to have two vectors **in** the plane than the vector **perpendicular** to the plane, so it is useful to have a standard method to calculate the vector perpendicular to two given vector. connecting the base point in the plane

$$\begin{aligned} u_0 a + v_0 b + w_0 c &= 0 \\ u_1 a + v_1 b + w_1 c &= 0 \end{aligned} \tag{$\perp$}$$

The cross product (part 2)

Given $\langle u_0 , v_0 , w_0 \rangle$ and $\langle u_1 , v_1 , w_1 \rangle$, we seek a $\langle a , b , c \rangle$ satisfying ($\perp$). **Eliminating** a from these equations gives

$$(u_1 v_0 - u_0 v_1) b + (u_1 w_0 - u_0 w_1) c = 0.$$

This (usually) determines b/c, and substituting in one of the original equations determines $\langle a , b , c \rangle$ up to a scalar multiple.

The cross product (part 3)

The solution

$$\langle a \, , \, b \, , \, c \rangle = \langle v_0 w_1 - w_0 v_1 \, , \, w_0 u_1 - u_0 w_1 \, , \, u_0 v_1 - v_0 u_1 \rangle$$

seems to be a **natural** scaling of the solution of $(\perp)$. The other solutions are scalar multiples of this vector. This construction defines the **cross product**.

Properties of the cross product

Since the terms of the cross product contain entries of each vector to the first power, it is easy to show that this product is **distributive**

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

This product is not commutative; instead we have

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

which has the unusual consequence that $\mathbf{a} \times \mathbf{a} = 0$ for all $\mathbf{a}$.

It is not useful to look at cross products of more than two terms.

Area

The length of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram with **a** and **b** as sides.

If you are interested in the area of a **triangle** with these sides, you need to multiply the length of the cross product by $\frac{1}{2}$.

More on the equation of a plane

Given three points A, B and C, there is (usually) a unique plane containing them. To find it, form the **vectors** $\vec{AB}$ and $\vec{AC}$ to get two directions in the plane. The cross product of these vectors is perpendicular to the plane.

This fails to find the equation of a plane is the cross product is zero, but this only happens when A, B, and C lie on a **line**.

Examples

#25: The plane through $(4, -2, 3)$ parallel to $3x - 7z = 12$.

#29: The plane through $(3, -1, 2)$, $(8, 2, 4)$, and $(-1, -2, -3)$.

#35: Find the point where the line $x = 1 + t$, $y = 2t$, $z = 3t$ meets the plane $x + y + z = 1$.

Tangent lines revisited

The tangent line to the graph of a function was central to many of the applications of single variable calculus. One way to express th property of the tangent is Taylor's formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2.$$

term is an **error term** giving the difference between the value $f(x)$ on the given curve and the y coordinate of the point on the tangent line for the same x. The ξ in this formula is a value between x_0 and x whose **existence** is asserted by Taylor's theorem although no attempt is made to find it. Instead, one uses its rough location to argue that $|f''(\xi)|$ is not too large.

Tangent lines revisited, part 2

When $|x - x_0|$ is small, this error term is not just the smallest term in the expression, but **much** smaller than the other terms. This says that the function may be reasonably well approximated by the tangent line in some interval around x_0. The tangent lines of space curves met in Chapter 13 have similar properties although proofs look a little different because space curves, including lines in space are defined parametrically.

Tangent planes

The definition of the tangent plane will require that one plane approximates the surface near a point uniformly in all directions. Some surfaces that are otherwise well-behaved, like the cone $z^2 = x^2 + y^2$, fail to have such a tangent plane at the origin. Although it has many planes with some of the properties of a tangent, it is better not to try to weaken the definition to allow more tangent planes. The strict requirement has so many useful consequences, and is satisfied in many cases, that little is lost be leaving a few examples out of the theory.

Derivatives and tangent planes

The geometric version of the existence of a derivative of a function f at a point (a, b) is the existence of a tangent plane to the surface $z = f(x, y)$ at the point where $x = a$ and $y = b$. A tangent plane has an equation of the form $z = Ax + By + C$ for constants A, B and C, which we abbreviate $z = L(x, y)$ — L standing for **linear**. The definition giving the most efficient characterization of tangent planes is to require that, for all $\epsilon > 0$,

$$|f(x, y) - L(x, y)| < \epsilon\sqrt{(x-a)^2 + (y-b)^2}$$

for all (x, y) sufficiently close to (a, b), independent of direction.

Derivatives and tangent planes, part 2

The expression on the right is chosen so that given any nonzero linear expression L, it must fail to bound L close to (a, b) for some ϵ. This allows the proof of the main theorem to be modified to show that there is at most one such L.

Using tangent lines to find tangent planes

Intersecting the tangent plane to $z = f(x, y)$ with the plane $y = b$ gives a curve whose equation is $z = f(x, b)$ and a tangent line to this curve at the point where $x = a$ in that plane. Since tangent lines could have been characterized by the same ϵ-δ definition as tangent planes, these tangent lines **must** lie in the tangent plane. Finally, writing the equation of the tangent plane in the form

$$z = L(x, y) = A(x - a) + B(y - b) + C$$

we can use what we know about tangent lines to show that $C = f(a, b)$, A is the derivative of $f(x, b)$ with respect to x evaluated at $x = a$, and B is the derivative of $f(a, y)$ with respect to y evaluated at $y = b$.

Exercises 14.4

Find equation of tangent plane at indicated point.

$$z = y^2 - x^2 \qquad (-4, 5, 9) \tag{1}$$

$$z = \sqrt{4 - x^2 - 2y^2} \qquad (1, -1, 1) \tag{3}$$

$$z = \ln(2x + y) \qquad (-1, 3, 0) \tag{5}$$

$$z = e^x \cos(xy) \qquad (0, 0, 1) \tag{13}$$