The definition of an integral

The definition of the integral in single variable calculus in terms of Riemann sums serves several purposes.

From the point of view of **pure mathematics**, it leads to a theorem that asserts that every continuous function can be integrated.

From the point of view of **numerical analysis**, the definition indicates that the integral can be approximated by a finite sum.

and . . .

The definition of an integral, part 2

From the point of view of **physics**, the integral is seen to provide a measurement for ordinary objects that agrees with what would be obtained if the body were approximated by a large number of point masses.

The definition of an integral in the plane

In order to have all of these viewpoints available for multiple integrals, a double Riemann sum of a function $f(x, y)$ over the rectangle

$$\mathcal{R} = \left\{ x, y \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d \right\}$$

is defined by partitioning $\mathcal{R}$ into rectangles of small diameter. The integral is allowed to exist if all of the sums give almost the same answer when the little rectangles are forced to have small diameters.

In particular, functions that are **discontinuous along curves** are integrable. However, from the point of view of **Calculus**, it is better to work with **continuous** functions on **general** domains.

Rectangles are dull

In the one variable case, the fact that continuous functions are integrable suffices for many examples, and is easily extended to show that **piecewise** continuous functions are integrable. In the several variable case, even such a simple problem as finding the area of a circle requires that the function that is 1 inside the circle and 0 outside be integrable. The key to including this in the theory is that boundary can be covered by rectangles of arbitrarily small total area. This is only a technical device allowing a special definition to be extended to include more useful cases. It has very little connection with the way in which we do calculus with multiple integrals.

Type I regions

For doing calculus, the basic regions are those for which the intersection with a vertical line is either empty or a single interval. The endpoints of this interval will depend on x. The text calls this “Type I” and writes the description as

$$\left\{ (x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x) \right\}.$$

Some examples are: (1) the **unit circle** with center $(0, 0)$,

$$a = -1, b = 1,$$
$$g_1(x) = -\sqrt{1 - x^2}, g_2 = \sqrt{1 - x^2};$$

More Type I regions

(2) the **triangle** with vertices $(0,0)$, $(x_0, 0)$, $(0, y_0)$,

$$a = 0, b = x_0, g_1(x) = 0, g_2(x) = y_0\left(1 - \frac{x}{x_0}\right);$$

(3) the area **below the line** $y = x + 2$ and **above the parabola** $y = x^2$,

$$a = -1, b = 2, g_1(x) = x^2, g_2(x) = x + 2.$$

Calculus books would say "between the line and the parabola", but extra detail (usually obtained from a graph) is needed when setting up the integral.

Type II regions

Reversing the roles of x and y leads to a region of “Type II”. Still more general regions can be obtained as unions of regions of these type that intersect only along curves.

It should be noted that Maple allows regions of these types to be described in its **plot3d** commands. The syntax of the descriptions of such regions is the same as that used in writing integrals over the region.

Iterated integrals

Once a description of a Type I region has been found, the integral of a function $f(x, y)$ over the region is given by

$$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right) dx.$$

The inner integral, with respect to y is formed treating x as a constant. For each x this is a definite integral, so all dependence on y will disappear when this is evaluated. The resulting expression may still depend on x, but when it is integrated with respect to x between constant limits, the answer will be a real number.

Iterated integrals, part 2

In writing such integrals, it is customary not to write the parentheses, but the limits on the **first integral sign** are associated with variable in the **last differential factor**. Such a pair serves to delimit the expression being integrated. This variable is also **available as a parameter** in describing limits of integration and the integrand of any included integral.

Exercises 15.3

Integrate $f(x, y)$ over $\mathcal{D}$.

#7. $f(x, y) = x^3 y^2$,

$$\mathcal{D} = \{ (x, y) : 0 \leq x \leq 2, -x \leq y \leq x \}.$$

#9. $f(x, y) = 2y/(x^2 + 1)$,

$$\mathcal{D} = \left\{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x} \right\}.$$

More Exercises 15.3

#15. $f(x, y) = y^3$, $\mathcal{D}$ is the triangle with vertices (0, 2), (1, 1), (3, 2).

#17. $f(x, y) = 2x - y$, $\mathcal{D}$ is the circle of radius 2 with center at the origin.

#23. $f(x, y) = 1 - x - y$, $\mathcal{D}$ bounded by $x = 0$, $y = 0$, $1 - x - y = 0$ (described as finding a volume).

Integration with respect to area

Although the integrals are written as

$$\iint_{\mathcal{D}} f(x, y)\,dx\,dy$$

to be ready for calculation as an **iterated integral**, the theory is based on identifying $dx\,dy$ as dA — the **element of area**. This will be important in applications, but it also plays a role in determining the changes necessary to integrate over regions defined in different coordinate systems.

Applications of double integrals

An easy way to claim that an integral arises as an application is to say that the integrand represents a **density**. This is mostly a matter of definition: if matter is spread unevenly over a plane region, it is reasonable to find an **average amount** over a piece of the region by cutting out the piece, weighing it, and dividing the weight by the area of the piece. If the piece is small enough, this will be close to a quantity that is **weight per unit area at a point**. It is more accurate physically to treat the weight as an indirect measurement of **mass**. The same approach could be applied to other forms of physical **content**, like **electrical charge**, but I will use mass for most of the examples.

Integration with respect to area, part 2

It is a form of the fundamental theorem of calculus that the total content of the region is the integral of density. Thus, if there is some way in which the density is known, integration recovers to total content of a region.

The density that is most likely to be known in advance is a **uniform** density. Then, mass is simply the product of area and density. The methods of single-variable calculus were often powerful enough to compute areas with a plausible argument that the integral was computing the intended area. From the point of view of this course, this has the unfortunate consequence that exercises with other density function seem artificial.

Moments

Even in the case of uniform density, double integrals are much easier to justify, since they are based on a more refined type of Riemann sum, and they allow related computations to be formulated using a unified argument. In particular, the **moment** of a planar object with respect to a line is is found by integrating the **signed distance** to that line with respect to mass. In single-variable calculus, this was done when the line was one of the coordinate axes, but any such integral can be easily formulated and evaluated as a double integral. The textbook is a little timid about approaching this topic and continues to emphasize moments with respect to the coordinate axes.

More Moments

There are also **second moments**, for which the integrand is the **square** of the distance to a line (or a point). The physical term for a second moment is **moment of inertia**. One benefit of having simple expressions for these quantities is that one can prove properties of the moments by considering properties of the quantity that is integrated to obtain that moment.

Theorems about first moments

Consider a region $\mathcal{R}$ represent a physical object in the plane whose total mass is M.

1. If L_1 is the line at signed distance d from line L_0, and the moment of $\mathcal{R}$ about L_i is denoted μ_i, then

$$\mu_0 = \mu_1 + Md$$

1a. Given L_0, one can find a parallel line L_1 such that $\mu_1 = 0$.

2. If the moments of $\mathcal{R}$ about two different lines through a point P are zero, then the moment of $\mathcal{R}$ about any line through P is zero. The point P is called the **center of mass** of $\mathcal{R}$.

Theorems about second moments

3. The second moment of $\mathcal{R}$ about a line L is always nonnegative, so it can be denoted σ^2.

3a. If L_0 passes through the center of mass of $\mathcal{R}$ and L_1 is parallel to L_0 at distance d, and if the second moment of $\mathcal{R}$ about L_i is σ_i^2, then

$$\sigma_1^2 = \sigma_0^2 + d^2.$$

4. The second moments about three different lines through the center of mass determine the second moments about all lines. There is an ellipse having the same second moments as $\mathcal{R}$.

Using previously calculated results

A sufficiently general calculation of the center of mass of a figure of a particular shape gives a result which has a geometric interpretation. A few of these are worth remembering. In exams in previous course, knowing the answer was mainly used as a check since exam questions were likely to emphasize setting up the integral and showing some details of its calculation. However, we now accept that you have **learned the calculus**, so there is **little reason to do** the these calculations. Indeed, a poor choice of coordinates can give you an integral that is not easy to evaluate even though it has a simple value. In this course, **you are encouraged to use previously computed results where appropriate**.

Examples of known moments

Example: Center of mass of a triangle.

With **uniform density**, the center of mass of a triangle is at the **centroid**, which can be found by averaging vectors from the origin to the three vertices.

If you know what the triangle looks like, you probably also have an easy way to find its area. Multiplying the area by the coordinates of the centroid gives the first moments.

Thus, the value of the integral of any linear function over the triangle can be found **without even setting up the integral**.

Area and Centroid of a triangle, part 1

To illustrate the value of a well-chosen coordinate system, consider the problem of finding the area an centroid of a triangle $\mathcal{T}$. Suppose the triangle is located so that on vertex is on the x axis and the opposite side is the line $y = h$ (as usual, h stand for the **height** of the triangle. This line will be taken as **base** with vertices at $(0, h)$ and (b, h). Then

$$A = \iint_{\mathcal{T}} dx\, dy \qquad \text{and} \qquad A\bar{y} = \iint_{\mathcal{T}} y\, dx\, dy.$$

(You will see why we don't compute $A\bar{x}$ this way.)

Area and Centroid of a triangle, part 2

In both cases, the first integral will be performed with respect to x. Since the integrand is a constant in these cases, the value is just the integrand times the length of the segment with second coordinate y. By properties of similar triangles, this is by/h. Thus

$$A = \int_0^h by/h\,dy = bh/2 \qquad \text{and} \qquad A\bar{y} = \int_0^h by^2/h\,dy = bh^2/3.$$

Area and Centroid of a triangle, part 3

This gives the usual formula for the area and tells us that the centroid lies on a line parallel to each side 2/3 of the way from the opposite vertex to that side. The point whose coordinates are the average of the coordinate of the vertices has this property for all three sides, so this must be the centroid.

Another approach would be to find the moment with respect to a median. By combining the previous results for the triangles on either side of the median, one finds that the moment about a median is zero, so the intersection of any two (and hence of all three) medians is the centroid.

Simple regions, part 1

We are now ready to give a form of the fundamental theorem of calculus relating double integrals over regions in the plane with line integrals around the boundary of the region. There is an annoying complication that arises when trying to state such a theorem: double integrals were described in an unoriented form useful for finding areas, moments and other physical quantities, but line integrals were given with an orientation that was a natural consequence of the use of a parameterization to reduce their evaluation to the familiar one-dimensional definite integral.

Simple regions, part 2

Setting up a double integral usually required a picture to check that all parts of the region were counted exactly once. The regions that we met were usually those described as “Type I”, with a description of the form

$$\mathcal{R} = \left\{ (x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x) \right\}.$$

This says that a vertical line $x = c$ that meets $\mathcal{R}$, meets it in an interval, and it meets $\mathcal{R}$ only for an interval of values of c.

Simple regions, part 3

The formalism serves only to provide names to the endpoints of these intervals. Integrals over more complicated regions can be found by writing the region as a union of Type I regions that intersect only in line segments, so that the integral over the whole will be the sum of integrals over the pieces.

Similarly, there were regions of “Type II” in which horizontal lines met the region in intervals.

Simple regions, part 4

A **simple** region is something that is of both Type I and Type II. A typical example is the portion of the first quadrant under the graph of a decreasing function. A common problem involving simple regions is to be given an integral using the Type I description of the region that looks difficult to evaluate, and to convert it to the equivalent integral using the Type II description of the region. This equivalent form turns out to be easy to evaluate, so you do it.

Proof of Green's theorem, part 1

Although we have not yet stated the theorem, we have all that we need to prove it.

The evaluation of the integral of $f(x, y)$ over our region $\mathcal{R}$ of Type I as an iterated integral proceeds by finding a function $F(x, y)$ with $F_2 = f$ (i.e., F is a **partial integral** of f with respect to the second variable) so that

Proof of Green's theorem, part 2

$$\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy = F\big(x, g_2(x)\big) - F\big(x, g_1(x)\big).$$

Then

$$\iint_{\mathcal{R}} f(x, y)\, dx\, dy = \int_a^b F\big(x, g_2(x)\big)\, dx - \int_a^b F\big(x, g_1(x)\big)\, dx.$$

Proof of Green's theorem, part 3

To get our proof, we note that the integrals on the right are of the form

$$\int_{\mathcal{C}} F(x, y)\, dx,$$

where $\mathcal{C}$ is a curve forming part of the boundary of $\mathcal{R}$. The boundary of $\mathcal{R}$ may also include line segments lying along $x = a$ or $x = b$, but these have no contribution to an integral with a dx factor. We may insert a multiple of such integrals to get a geometrically satisfying result without changing the value of the integral.

Proof of Green's theorem, part 4

Similar integrals with respect to y are found by using a Type II description to integrate $f(x, y)$ with respect to x.

We need only interpret these results to get Green's theorem.

The boundary of a region, part 1

Consider the case in which $\mathcal{R}$ is the unit circular disk defined by

$$\left\{ (x, y) : x^2 + y^2 \leq 1 \right\}.$$

The boundary is the circle with equation $x^2+y^2 = 1$, which we always parameterize by $x = \cos t$ and $y = \sin t$. In this parameterization, we go around the circle in a counterclockwise direction, and the disk is always to our left.

Something similar can be done for every simple region. To keep the region to our left, we should go from $x = a$ to $x = b$ along the bottom of the region, then up along $x = b$, then from $x = b$ to $x = a$ along the top, and finally down along $x = a$.

The boundary of a region, part 2

Note that, if a Type I region is cut along the graph of a function, that curve would be traversed from left to right as the bottom of the upper part of the region and from right to left as the top of the lower part of the region. In this way, we see that a line integral over the boundary of the union of the two pieces is the sum of line integrals over the boundaries of the two pieces.

The boundary of a region, part 3

If $\mathcal{C}$ is a closed curve, bounding a region $\mathcal{D}$, we use

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

to denote an integral that goes once around $\mathcal{C}$ in the direction that keeps $\mathcal{D}$ to the left. For simple regions, this direction on $\mathcal{C}$ is easy to find. This construction extends to regions that can be cut into simple parts. At some point, one should show that the oriented boundary of a region does not depend on how the region is cut into simple pieces. The regions for which this gives a well-defined oriented boundary exhaust all regions that you are likely to meet, although the full mathematical generality allows regions that can be approximated by such regions.

A statement of Green's theorem

If $\mathcal{D}$ is a simple region with boundary $\mathcal{C}$, then

$$\oint_{\mathcal{C}} P(x,y)\,dx = \iint_{\mathcal{D}} -P_2(x,y)\,dx\,dy$$
$$\oint_{\mathcal{C}} Q(x,y)\,dy = \iint_{\mathcal{D}} Q_1(x,y)\,dx\,dy$$

Note that the choice of orientation of the boundary leads to the pattern of signs in these formulas.

Formula (1) in Section 16.4 results from adding these two formulas.

Conservative vector fields

The test for $\langle P, Q \rangle$ being conservative is $Q_1 - P_2 = 0$. Green's theorem thus includes the statement that the integral of a conservative vector field around the boundary of any region is zero. One needs these derivatives to exist **everywhere** inside the region bounded by the curve for this proof to work. We will give an example where the derivatives fail to exist at a single point and the integral is nonzero along closed curves enclosing that point.

Using Green's Theorem

The typical exercise in the use of Green's theorem involves evaluation of line integrals around closed curves by transforming them into double integrals. Of course, the proof of Green's theorem shows that the evaluation of the double integral involves replacing it by a line integral — but the resulting line integral is often much simpler than the one you started with. The two line integrals related by this operation differ by the integral of a conservative vector field.

The examples obtained by interchanging order of integration in simple regions also have first steps that give line integrals of vector fields whose difference is conservative.

Not so simple regions

After performing detailed calculations on simple regions, the results can be applied to more general regions by formal means. The main tool cuts along a straight line. The integrals associated with the whole region are easily seen to be the sum of corresponding integrals for the parts.

Since line integrals are simpler from the theoretical point of view, they can be used to interpret what the corresponding double integrals must be for Green's theorem to remain true. This leads to an interpretation of oriented and multiple areas. Although this makes the double integral more difficult to interpret, it leads to simpler rules for manipulating them, that is to a **calculus**.