What is Green’s Theorem, really?

The identity expressed by Green’s theorem **looks significant**, but what is it good for? Textbook exercise concentrate on a superficial use of the formula to convert an integral over a region to one around its boundary, or *vice versa*, but they give no hint why one would ever need to do this. These lectures started with the **proof** of the theorem to remind you that the usual calculation of double integrals as **iterated integrals** is a form of Green’s theorem.

What is Green's Theorem, really? (part 2)

For **simple regions** the useful trick of interchanging order of integration produces line integrals that look different, but give the same answer because both are equal to the same double integral. The connection by way of Green's theorem looks more complicated because **orientation** is usually hidden in geometric arguments based on a picture when working with double integrals, but it is an **essential part** of Green's theorem.

The line integrals in Green's theorem are **closed curves** that follow the boundary of the region, **keeping the interior of the region to the left**. For **convex** regions, this means going around the region in a counterclockwise direction.

What is Green's Theorem, really? (part 3)

When orientation is taken into account, the line integrals giving the area of a region are $\oint x\,dy$ or $\oint -y\,dx$. In the same way, the volume of a figure of rotation (using an rh coordinate system with the h axis being the axis of rotation) are $\oint \pi r^2\,dh$ or $\oint -2\pi rh\,dr$.

A benefit of considering this application to be part of Green's Theorem is that the ability to choose between these line integrals has been made independent of the shape of the regions. It is not necessary to have a Type I region in order to express a double integral in the form $\oint P\,dx$. Such an integral can be evaluated and will give the right answer as long as some parameterization of the boundary is available.

The whole is the sum of the parts

Another benefit of considering the change of order of integration as a part of Green's Theorem is that there is no obligation to do the same thing to all parts of the integrand. If it seems more natural to integrate one part of the integrand with respect to x and another part with respect to y, nothing stops you from doing this to create a vector field whose integral around the boundary evaluates the given double integral.

The whole is the sum of the parts, part 2

For example, integrating

$$\frac{1}{2}\left(x\,dy - y\,dx \right)$$

around the boundary of a region gives the area of the region. When combined with a parametrization of the boundary, this is often easier than either of the traditional calculations of area. The case of the unit circle is an impressive example.

Polar coordinates

The equations $x = r\cos\theta$ and $y = r\sin\theta$ relate the rectangular coordinates x and y to the polar coordinates r and θ. The periodicity of the trigonometric functions shows that adding 2π to θ and using the same r gives the same values of x and y. A **slightly closer look** at the trig functions shows that adding π to θ and multiplying r by (-1) also gives the same x and y. This operation shows that the circle represented by the polar equation $r = \cos\theta$ is completely traversed when θ goes through an interval of length π.

Polar coordinates, part 2

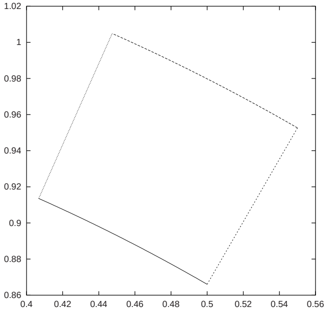
Since points in the xy-plane have infinitely many representations in polar coordinates, it will be important when constructing integrals to know how to produce inequalities in r and θ that represent **only one** set of coordinates for the points in a given region. This will require some care, but it is not particularly difficult if you allow yourself to be guided by a picture.

There are a small number of curves that give the common examples of curves given by polar equations. You should do enough exercise to meet them all.

Polar rectangles

A set given by the conditions $a \leq r \leq b$ and $\alpha \leq \theta \leq \beta$ is called a **polar rectangle**.

Here is an example.



Polar Riemann Sums

The definition of the double integral can be copied for polar coordinates by partitioning a portion of the plane into small polar rectangles, i.e., polar rectangles in which both $b - a$ and $\beta - \alpha$ are small. Although slightly bent, the sides of a polar rectangle are seen to have lengths Δr and $r\,\Delta\theta$ and an area $r\,\Delta r\,\Delta\theta$. In fact, a polar rectangle can be seen to have **exactly** this area with $r = (a+b)/2$.

The double integral of the function $f(x, y)$ over a region $\mathcal{R}$ in the xy-plane was defined as a limit of sums over small rectangles of the value of f somewhere in the rectangle times the area of the rectangle. This is often interpreted as the volume of a figure with base $\mathcal{R}$ whose height at (x, y) is $f(x, y)$.

Polar Riemann Sums, part 2

The only expression in polar coordinates that has a chance of representing the same thing is a sum over polar rectangles of $f(r\cos\theta, r\sin\theta)$ for values of r and θ within the intervals specifying the polar rectangle times **the area of the polar rectangle**, which we know to be $r(\Delta r)(\Delta\theta)$ for a value of r somewhere inside the polar rectangle.

The element of area

This approach to integration in polar coordinates is based on duplicating the definition of integrals using Riemann sums. If you don't believe that Riemann sums are useful, this isn't completely satisfactory. In later sections, we will see other ways to explain that the equation

$$dA = dx\,dy = r\,dr\,d\theta$$

gives polar integrals that agree with the integrals defined in rectangular coordinates. However, the underlying geometry makes it clear that this is the only computation that has a chance of giving the right answer.

The "Change of Variables" Formula

If we want to calculate

$$\iint_{\mathcal{R}} f(x, y)\, dx\, dy$$

using polar coordinates, we must form

$$\iint_{\mathcal{R}^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, r\theta,$$

where $\mathcal{R}^*$ stands for the description in polar coordinates of $\mathcal{R}$. In exercises in this course, the rectangular description $\mathcal{R}$ and the polar description $\mathcal{R}^*$ are usually found using a picture of $\mathcal{R}$ which is interpreted in polar coordinates. The picture is also used to assure that everything has been represented so that $dx\, dy$ and $r\, dr\, d\theta$ represent positive fragments of area.

The usual order of integration

In contrast to rectangular coordinates where "Type I" and "Type II" descriptions must often both be studied before deciding how to calculate an integral, polar integrals are almost always calculated by integrating first with respect to r. This is due to polar coordinates being used primarily to describe curves in which r is an explicit function of θ.

To find the area inside a polar curve $r = f(\theta)$ using double integrals, we form

$$\int_{\theta_0}^{\theta_1} \int_0^{f(\theta)} r \, dr \, d\theta = \int_{\theta_0}^{\theta_1} \frac{1}{2} f(\theta)^2 \, d\theta$$

in agreement with the formula introduced in Section 10.5.

The usual order of integration, part 2

Since the double integral gives the previous formula **so easily** using one order of integration, there is no reason not to formulate **all** problems as multiple integrals instead of relying on the earlier formulas for getting single integrals. Multiple integrals **always** have the advantage of allowing additional methods of evaluation which may lead to simpler calculation.

Exercises

Find the integral

$$\iint_{\mathcal{R}} f(x, y)\, dA.$$

#7 $f(x, y) = x$, $\mathcal{R}$ is the disk of radius 5 centered at the origin.

#9 $f(x, y) = xy$, $\mathcal{R}$ is the region in the first quadrant between $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.

#11 $f(x, y) = e^{-x^2-y^2}$, $\mathcal{R}$ is the semicircle bounded by $x = \sqrt{4 - y^2}$ and the y-axis.

More Exercises

#13 $f(x, y) = x^2 + y^2$, $\mathcal{R}$ is the region bounded by the spirals $r = \theta$ and $r = 2\theta$ for $0 \le \theta \le 2\pi$

#15 $f(x, y) = 1$ (finding area), $\mathcal{R}$ is one loop of the rose $r = \cos 3\theta$.

#23 Find volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2+y^2+z^2 = 1$.

#27 Evaluate by converting to polar coordinates:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx$$