A key example

If

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle,$$

then it appears that the condition for **independence of path** is satisfied. However, $\mathbf{F}$ and its derivatives fail to be defined at $(0, 0)$. Although Green's theorem continues to hold for any path that **does not enclose the origin**, we shall see that this one troublesome point is enough to cause $\mathbf{F}$ to fail to be conservative.

It is easy to compute its integral around the unit circle, $x = \cos t$, $y = \sin t$, from $t = 0$ to $t = 2\pi$. Here, $dx = -\sin t\, dt$ and $dy = \cos t\, dt$, so $-y\, dx + x\, dy = dt$.

A key example, part 2

The integral once around the unit circle in the positive direction is 2π.

The value of the integral we have given on any path not containing $(0, 0)$ is always an integer multiple of 2π. That integer, called the **winding number**, can be interpreted as the number of times the path goes around the origin in the counterclockwise sense (so that it is -1 if you go once around in the clockwise sense).

Using Green's Theorem for change of coordinates, part 1

The proof of Green's theorem shows that every double integral in the xy can be evaluated as a suitable line integral around its boundary. If x and y are functions of u and v, this is converted to a line integral in the uv plane. This line integral is converted by Green's theorem into a double integral in the uv plane. So far, we have done this only for polar coordinates, and no separate r, θ plane, in which these are **rectangular** coordinates was introduced. However, the proof via Green's Theorem needs this plane for a **second** application of Green's Theorem. In this proof, we can start with a line integral

$$I = \oint P\,dx + Q\,dy.$$

Using Green's Theorem for change of coordinates, part 2

Here, P and Q will be functions of x and y. We then express x and y in terms of u and v, which will be expressed in terms of u and v by composition with the functions giving x and y in terms of u and v. This substitution will express I as a line integral in the uv-plane. In addition to obtaining P and Q in terms of u and v, we need to write

$$dx = x_u\,du + x_v\,dv,$$

$$dy = y_u\,du + y_v\,dv.$$

Thus,

$$I = \oint (Px_u + Qy_u)\,du + (Px_v + Qy_v)\,dv.$$

Using Green's Theorem for change of coordinates, part 3

Applying Green's theorem in the uv-plane gives an integrand of the double integral that is

$$
\begin{aligned}
(Px_v + Qy_v)_u - (Px_u + Qy_u)_v = \\
Px_{vu} + Qy_{vu} + (P_x x_u + P_y y_u)x_v \\
+(Q_x x_u + Q_y y_u)y_v \\
-Px_{uv} - Qy_{uv} - (P_x x_v + P_y y_v)x_u \\
-(Q_x x_v + Q_y y_v)y_u \\
= (Q_x - P_y)(x_u y_v - x_v y_u)
\end{aligned}
$$

Using Green's Theorem for change of coordinates, part 4

Here the first part is the integrand of the original double integral in the xy plane composed with the definition of x and y in terms of u and v, and the second factor is the ratio of areas in the two planes. This factor is called the **Jacobian**. In polar coordinates, it reduces to r.

There are some other changes of coordinates that can be used for simple examples. One is $x = u/v$, $y = uv$ in the first quadrant. The Jacobian is the determinant of

$$\begin{bmatrix} 1/v & -1/v^2 \\ v & u \end{bmatrix}$$

which is $2u/v$.

Using Green's Theorem for change of coordinates, part 5

The "rectangles" in this set of coordinates are lines through the origin that have constant y/x and the hyperbolas that have constant xy. The order of coordinates is significant here since this choice gives a positive Jacobian in the first quadrant, corresponding to agreement between the positive orientations of the coordinate vectors.

Calculus or Advanced Calculus?

From the point of view of this course, the main use of this formula involves integrals over regions described in polar coordinates, although it provides a simple way to evaluate integrals in coordinate systems that were **invented** to simplify particular integrals. All of these integrals could be done almost as easily using the **parameterization** suggested by the special coordinate system applied to **any suitable vector field** around the boundary of the region. Simple exercises should remain simple however they are done.

Calculus or Advanced Calculus?,part2

More advanced work requires that all aspects of **Vector Calculus**, as it would be interpreted in a rectangular coordinate system, be expressed in the new coordinates. The VectorCalculus package in Maple allows this to be done automatically, although we do not use this ability in order to concentrate on the fundamental meaning of the operations and the theorems that illustrate it in Cartesian coordinates.