

The setting for Stokes' Theorem

It is conventional to state theorems before proving them, but this sometimes leads to unmotivated work establishing the definitions needed to state the result. Since **the motivation lies in the proof**, we give the proof first, and then interpret it. We use (x, y, z) for the coordinates in the \mathbb{R}^3 where all objects are constructed.

One part of the theorem concerns a **surface** \mathcal{S} that we begin by assuming to be **the graph of a function** $z = g(x, y)$. This suffices for our needs since we need only have enough of a proof to provide clues to the correct statement of the general theorem and interpretations of the formulas it relates.

The setting for Stokes' Theorem, part 2

Another part of the theorem requires a **closed curve** \mathcal{C} **lying in the surface** \mathcal{S} . As usual, \mathcal{C} is assumed to be **given by a parameterization**

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Since \mathcal{C} **is contained in** \mathcal{S} , we must have $z(t) = g(x(t), y(t))$. Also, let \mathcal{C}_0 be the **projection** of \mathcal{C} into the xy -plane, so that it is parameterized by $\langle x(t), y(t) \rangle$.

The line integral in Stokes' Theorem

One side of the Stokes Theorem equation is the **integral of a vector field** \mathbf{F} around \mathcal{C} .

Write the **vector field** as

$$\mathbf{F} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle.$$

The line integral in Stokes' Theorem, part 2

Then $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is an abbreviation for the expression

$$\int_a^b P(x(t), y(t), z(t)) x'(t) + \\ Q(x(t), y(t), z(t)) y'(t) + \\ R(x(t), y(t), z(t)) z'(t) dt,$$

where the values $t = a$ and $t = b$ correspond to **going once around** \mathcal{C} in the counterclockwise direction when viewed from above.

Projection

We next write expressions for the interpretations of these integrals as **line integrals on \mathcal{C}_0** . This involves replacing every mention of z by $g(x, y)$. **This is straightforward** in the first two terms, but in the third term,

$$z'(t) = g_1(x(t), y(t)) x'(t) + g_2(x(t), y(t)) y'(t).$$

where g_i indicates the **partial derivative of the function** g with respect to its i^{th} argument. Omitting **explicit mention** of t , the integral can be written as a **line integral**

Projection, part 2

$$\oint_{\mathcal{C}_0} (P(x, y, g(x, y)) + R(x, y, g(x, y)) g_1(x, y)) dx + \\ (Q(x, y, g(x, y)) + R(x, y, g(x, y)) g_2(x, y)) dy.$$

This line integral in the xy plane is equal, by **Green's Theorem**, to the **double integral over the region** \mathcal{S}_0 bounded by \mathcal{C}_0 of

$$\frac{\partial}{\partial x} (Q(x, y, g(x, y)) + R(x, y, g(x, y)) g_2(x, y)) - \\ \frac{\partial}{\partial y} (P(x, y, g(x, y)) + R(x, y, g(x, y)) g_1(x, y)).$$

Simplifying the integrand

One term:

$$\begin{aligned} \frac{\partial}{\partial x} (Q(x, y, g(x, y)) + R(x, y, g(x, y))g_2(x, y)) = \\ Q_1(x, y, g(x, y)) + Q_3(x, y, g(x, y))g_1(x, y) + \\ R(x, y, g(x, y))g_{21}(x, y) + R_1(x, y, g(x, y))g_2(x, y) + \\ R_3(x, y, g(x, y))g_1(x, y)g_2(x, y) \end{aligned}$$

Simplifying the integrand, part 2

The other term

$$\begin{aligned} \frac{\partial}{\partial y} & (P(x, y, g(x, y)) + R(x, y, g(x, y))g_1(x, y)) = \\ & P_2(x, y, g(x, y)) + P_3(x, y, g(x, y))g_2(x, y) + \\ & R(x, y, g(x, y))g_{12}(x, y) + R_2(x, y, g(x, y))g_1(x, y) + \\ & R_3(x, y, g(x, y))g_1(x, y)g_2(x, y) \end{aligned}$$

Interpreting the planar integral

The terms containing second derivatives of g or products of two first derivatives of g in this expression cancel, and the remaining terms may be grouped as

$$(R_2 - Q_3)(-g_1) + (P_3 - R_1)(-g_2) + (Q_1 - P_2).$$

The second factors in these terms are the components of $\langle -g_1, -g_2, 1 \rangle$ which is **perpendicular to** \mathcal{S} . The first factor must then be the components of a **vector field** being integrated over the surface. Note that the normalization of our normal vector to have third coordinate 1 signifies upward orientation and integration with respect to $dx dy$.

Interpreting the planar integral, part 2

The other factor is a vector field constructed from derivatives of \mathbf{F} . This expression is often described as $\nabla \times \mathbf{F}$ since the pattern of partial derivatives follows the **same pattern as terms** in a cross product. This construction is called the **curl** of \mathbf{F} .

Other coordinate planes

The previous analysis applies on any piece of the surface where z **can be given** as a function of x and y . A computation of the surface integral using parameters other than x and y only requires a different **scaling** of the normal vector.

For example, the rightward orientation for an integral with respect to y and z would have first coordinate $+1$.

Orientation in coordinate planes

The classical xy plane with **horizontal** x -axis and **vertical** y -axis corresponds to the view of the plane from the positive z -axis. The signs that come from expanding determinants show that the corresponding views from the positive x -axis has a horizontal y -axis and vertical z -axis; and from the positive y -axis, a horizontal z -axis and vertical x -axis.

From particular to general

All of our oriented integrals are such that cutting the region into pieces gives the integral as the sum over the pieces. The **Implicit Function Theorem** tells us that a nonzero component of the normal at a point allows a small piece containing the point to be found on which the selected variable is a function of the remaining variables. Since **any** coordinate plane could have been used in our proof, the whole surface is thus broken into pieces covered by our special case.

Conservative vector fields

One observation connected to Stokes' Theorem is that the components of $\nabla \times \mathbf{F}$ are exactly the things that must be tested to show that \mathbf{F} is **conservative**. Since the line integral of a conservative field is zero around every curve, it is natural to expect that its corresponding surface integral would also be identically zero. As in the case of Green's Theorem, one can use Stokes' Theorem to replace a line integral by a surface integral to remove conservative vector fields that may complicate the computation without affecting the value of the integral.

Conservative vector fields, part 2

Although we will see that surface integrals usually depend on the surface, the surface integrals in Stokes' Theorem give the same value for integrals over any surfaces having the same boundary curve.

The significance of Stokes' Theorem

If \mathcal{C} is a closed curve, we have shown that

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot \mathbf{n} dS$$

for **any** surface \mathcal{S} bounded by \mathcal{C} , but how do we evaluate the integral?

If you have a parameterization of \mathcal{C} , the line integral is easy to evaluate directly. Sometimes $\nabla \times \mathbf{F}$ is much simpler than \mathbf{F} itself, so it may be tempting to evaluate the flux integral, but then we need \mathcal{S} .

The significance of Stokes' Theorem, part 2

If a surface isn't given isn't given (and even sometimes when it **is** given), it needs to be constructed from the description of \mathcal{C} . No single recipe is available for finding \mathcal{S} , but if you can find a surface, use it.

A striking property of Stokes' Theorem is that the integral doesn't depend on \mathcal{S} . This allows you to use the simplest admissible surface to compute the flux integral. It will usually be computed by projecting into a coordinate plane — as in the proof of the theorem — but a parameterization may be use if one is available.

The significance of Stokes' Theorem, part 3

It is also possible — as with Green's Theorem — that the line integral could be used to evaluate the flux integral. To do this, one would need to recover \mathbf{F} from its curl. The exercises in the text avoid this by **giving** \mathbf{F} . However, not every vector field is the curl of another vector field, so only special flux integrals are equivalent to line integrals.

Flux integrals

The integrals arising in Stokes' Theorem are **flux integrals**. That is, they extract the **normal component** of the vector field in the integrand by forming the **dot product** with a vector perpendicular to the surface.

While the **direction** of the normal vector is determined, it is scaled depending on the variables of integration. A natural choice would be to use a **unit** normal vector, and interpret the length as a **Jacobian** for a change of variables. Since the Jacobian is the **local ratio of area** in the two coordinate systems, the length of our original vector should be the area in the tangent plane that projects into a unit area in the xy -plane.

Flux integrals, part 2

The relations between cross products and area tells us that this is the area in the tangent plane. The notation $\mathbf{n} dS$ expresses this, where dS suggest an element of surface area.. However, this is used only for **talking about** the integral. **Surface area plays no role in the computation of the integral.** Computation is done by **projecting** into one of the coordinate planes, or by **parameterizing** the surface. We shall see that the integrals giving surface area are rarely computable in close form, but these **flux integrals** depend nicely on the equation of the surface.

Integrals of vector fields over surfaces, part 1

Many physical applications involve vector fields interpreted as **flows** and require the measurement of the **flux** through a surface. The contribution to the flux of a small piece of the surface should be the product of its area and the component of the flow perpendicular to it (since flow parallel to the surface does not cross it). There is an underlying assumption that the flow is a **vector** quantity and that it behaves in a **linear** fashion when vectors are added or multiplied by scalars. For anything with this behavior, it is essential that it be measured by a quantity that is sensitive only to normal components.

Integrals of vector fields over surfaces, part 2

These integrals are also **oriented**, in the sense that reversing the direction of the flow should give the negative of the previous measurement. This requires that the surfaces have a clear **inside** and **outside**. In many of the exercises, the surface is the graph of a function $z = f(x, y)$ and an **upward** direction (i.e., a positive third component) can be used as a substitute for “outward”.

There are surfaces that do not have a global orientation. That is, you can walk around the surface carrying a continuously varying unit normal vector and get back to the same point with the normal having reversed its direction.

Integrals of vector fields over surfaces, part 3

The Möbius band is a common example: a rectangle is given a half-twist before gluing its ends together, so points that were on one side of the surface now find themselves next to points that were originally on the other side. We won't do anything with such surfaces except to acknowledge their existence.

In a later section, we will use surface integrals around closed surfaces to measure properties of the region inside the surface. For such results to make any sense, the surface must **have an inside** — i.e., it must have a global orientation. This is an extra concern in formulating theorems in this area, but it turns out not to cause any real trouble.

Computing surface integrals, part 1

When you want to do calculus, you determine what the expression you are computing looks like in the special case of the graph of a function. Since we are now concerned with surfaces, this means $z = f(x, y)$. We have already noted that a consistent orientation can be provided by the notion of **upward**. It remains only to convert the flux integral

$$\iint \mathbf{F} \cdot \mathbf{n} \, dS$$

into something that we can compute.

Computing surface integrals, part 2

Although the notation suggests that we should find \mathbf{n} and dS separately and substitute our findings into the definition, these two terms should be considered as a single object. The reason for this is the formula

$$\frac{dS}{\sqrt{A^2 + B^2 + C^2}} = \frac{dx dy}{C} = \frac{dx dz}{B} = \frac{dy dz}{A}$$

The homogeneous nature of this formula means that $\langle A, B, C \rangle$ can be taken to be any normal vector.

Computing surface integrals, part 3

This isn't quite right since we are integrating a quantity that may change sign and we need some way to be sure that our answer has the correct sign. This is best dealt with by making the process completely formal, but it is difficult to appreciate the abstraction required until you have some experience where it is required.

If you multiply by $\langle A, B, C \rangle$, the first expression is exactly $\mathbf{n} dS$, and the others are the expressions to be used in computation. In one version, \mathbf{F} is written as $\langle P, Q, R \rangle$ and the separate terms are written in the simplest form to get

$$\iint P dy dz + Q dz dx + R dx dy. \quad (*)$$

Computing surface integrals, part 4

Another form which is suitable for graphs of functions used the vector with $C = 1$ as a normal to get

$$\iint \left(-P \frac{\partial z}{\partial x} - Q \frac{\partial z}{\partial y} + R \right) dx dy.$$

The connection between these expressions is given by replacing dz by its expression in terms of dx and dy and treating $dx dx$ or $dy dy$ as zero, and $dy dx$ as $-dx dy$.

Exercises, Section 16.7

Find $\iint \mathbf{F} \cdot \mathbf{n} \, dS$ over the given region (with upward orientation).

#19. $\mathbf{F} = \langle xy, yz, zx \rangle$ on $z = 4 - x^2 - y^2$ above $0 \leq x \leq 1, 0 \leq y \leq 1$.

#21. $\mathbf{F} = \langle xye^y, -xze^y, z \rangle$ on the portion of $x + y + z = 1$ in the first octant.

More Exercises, Section 16.7

An example using (*) is

#27. $\mathbf{F} = \langle x, 2y, 3z \rangle$ on the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ with outward orientation.

Another problem using unusual roles for the coordinate variables is

#25. $\mathbf{F} = \langle 0, y, -z \rangle$ with the outward orientation on the closed surface formed by the paraboloid $y = x^2 + z^2$ for $0 \leq y \leq 1$ and the disk $x^2 + z^2 \leq 1$ in the plane $y = 1$.