Triple integrals

The definition of triple integrals involves Riemann sums based on partitioning intervals on the three coordinate axes into small subintervals. Such a definition is made to assure that certain physical and geometric quantities are given by integrals, and to prove the existence of the integral of functions that are continuous except of sets of lower dimension. This allows the theory to be based on integrals over rectangles with the extension to more general regions provided by multiplying he expected integrand by a function that is 1 on the domain of interest and 0 elsewhere. This is a wonderful **theoretical tool**, but it is **never** used in **calculus**.

Iterated integrals

In this course, integrals are usually evaluated by using a geometric description of the region in which the section parallel to **one of the axes** is described. This works best when these intersections are always intervals. Then, the upper and lower endpoints are given as functions of the **other two variables** — where the intersection is nonempty. The set where there is a nonempty intersection is the **projection** of the three dimensional region into one of the **coordinate planes**. A similar description of such plane regions was given in the discussion of double integrals.

Review of double integrals

Double integrals were first mentioned in Sections 15.1–3 (Feb. 21). A brief mention of the use of Riemann sums was followed by the claim the **Rectangles are dull**, which was intended to discourage the expectation that constant limits of integration would appear in exam problems. Although this led to a good result on Problem 6 of Exam 2, it bears repeating. Techniques for setting up double integrals will be re-examined as the first step in dealing with triple integrals.

This method will be used **twice** in setting up triple integrals: once in constructing the section parallel to the first selected axis; and again in setting up a double integral over the projection.

Type I regions in the plane

The basic domains of integration for double integrals were those for which the intersection with a vertical line is either empty or a single interval. The endpoints of this interval will depend on x. This is an extension of the application of $\int_a^b f(x)\,dx$ as the area under the graph of $y = f(x)$ between the vertical lines $x = a$ and $x = b$.

A description of a Type I region has the form

$$\left\{ (x, y) : a \le x \le b, g_1(x) \le y \le g_2(x) \right\}.$$

which says that the test for a point (x, y) to be in the region consists of checking some **numerical** bounds on x, and then bounds on y that depend on x.

Type I regions in the plane, part 2

Implicit in this description is that $g_1(x) \leq y \leq g_2(x)$ for all x between a and b. Frequently, one or both of the bounds on x are consequences of this requirement. Geometrically, this is the case when the region has a **corner** rather than a **vertical side** at this value.

Example 1

The portion of the first quadrant inside the unit circle.

To check the description, we ask Maple to draw a picture. The worksheet for this lecture compares a direct graph of the region with the detailed description need for an integral.

Example 1, continued

First quadrant: (A) $x \geq 0$ and (B) $y \geq 0$.

Inside the unit circle: (C) $x^2 + y^2 \leq 1$.

Solve (C) for y to get $-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}$. Since $-\sqrt{1-x^2} \leq 0$ whenever it is defined, (B) and (C) give $0 \leq y \leq \sqrt{1-x^2}$.

Again $0 \leq \sqrt{1-x^2}$ whenever it is defined, which is for $-1 \leq x \leq 1$. Combining this with (A) gives $0 \leq x \leq 1$. All of (A), (B) and (C) have been used.

Integrals over this region

If you have the precise description of the region by inequalities, that information can be written in the notation for the integral in what should be seen as the **natural locations** to obtain

$$\int_0^1 \int_0^{\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$

Here, the function f is whatever function you want to integrate over the region. For example, $f(x, y) = 1$ if you want to compute the area of the quadrant, $f(x, y) = x$ if your want the first moment with respect to the y axis, or $f(x, y) = \sqrt{1 + 4x^2 + 4y^2}$ if you want the area of the portion of the paraboloid $z = x^2 + y^2$ above this region.

An important principle

In all applications, the limits of integration and integrand are determined independently. The limits of integration are the same for all applications with the same region, and the integrand is the same for all with the same physical or geometric interpretation.

Another example

Suppose we are told to integrate over the region **between** $y = x$ and $y = x^2$. It is common to begin with a picture, so we have produced one in Maple.

From the picture, it is easy to verify that the only **bounded** region between these curves is **above** $y = x^2$ and **below** $y = x$, leading to $x^2 \leq y \leq x$. The numerical limits on x are found by **solving the inequality** $x^2 \leq x$.

Evaluating iterated integrals

In single-variable calculus, it is possible to pretend that differentiation and integration are just operations on functions. The dx at the end of the notation for integrals seems like just something more to write in the initial examples. It becomes useful when **integrating by substitution**, but it is usually possible to minimize its role in most problems.

Evaluating iterated integrals, part 2

With double integrals, systematic use of these differentials forces a clear organization on the work. Everything between the initial integral sign with constant limits and the final dx is the description of the function of x to be integrated. In the case of a double integral, it is an integral with respect to y that can contain x as a parameter **everywhere**: in the limits of integration as well as in the integrand. Since the inner integral ends with dy, y is the only symbol that is considered a variable in that integral. In other words, this is a **partial integral** to be found by obtaining an expression whose partial derivative with respect to y is the given integrand.

Relation with Green’s Theorem

The first way of writing this partial integral is as a function of x and y. However, these integrals are **definite integrals** in which one must substitute the limits of integration for y to obtain a function of x alone.

The general process of finding $\int_a^b F(x, y)\,dx$ with the requirement that y be replaced by a certain function of x is called a **line integral**, and it corresponds to integrating along a curve.

Relation with Green’s Theorem, part 2

The proof of Green’s Theorem consists of identifying line integrals of partial integrals around the boundary of a region that are equal to a given double integral. After proving the theorem, it is possible to shift emphasis to show that a line integral around the boundary can be expressed as double integral over the interior of some kind of derivative of the expression in the line integral.

Advanced methods

It is possible to change variables in double integrals. An important example is given by the use of **polar coordinates**. At one level, this consists only of additional ways to discover **partial integrals** that express the double integral over a region as a line integral around the boundary. after all, the first step in evaluating an iterated integral is **always** to obtain an equivalent line integral. The integrals obtained through the use of polar coordinates are often simpler because of the use of symmetry or some other special role of the origin in the region.

Section 15.7, Exercise #3

Evaluate

$$\int_0^1 \int_0^z \int_0^{x+z} 6xz\,dy\,dx\,dz$$

The innermost integral is

$$\int_0^{x+z} 6xz\,dy = 6xzy\Big|_{y=0}^{x+z} = 6xz(x+z) = 6x^2z + 6xz^2$$

since the integrand is **constant** as a function of y.

Section 15.7, Exercise #3, part 2

Next, consider

$$\int_0^z 6x^2z + 6xz^2\,dx = 2x^3z + 3x^2z^2\Big|_{x=0}^{z} = 2z^4 + 3z^4 = 5z^4.$$

Finally, we have

$$\int_0^1 5z^4\,dz = z^5\Big|_{z=0}^{1} = 1.$$

Section 15.7, Exercise #7

Evaluate

$$\iiint_{\mathcal{E}} 2x\, dV$$

where

$$\mathcal{E} = \left\{ (x, y, z) | 0 \le y \le 2, 0 \le x \le \sqrt{4 - y^2}, 0 \le z \le y \right\}$$

These inequalities are easily translated into Maple instructions for drawing the faces of this body, and they are in the worksheet for this lecture.

Evaluation of Exercise #7, part 2

Since y has constant limits, that should be the outer integral. Either x or z could be specified next since both have bounds depending only on y. We take z next and x as the innermost integral. Details are in the Maple worksheet.

Special integrands

Integrating the constant 1 over a region gives **volume**. In the special case where we are considering the regions under the graph of $z = f(x, y)$, integrating with respect to x gives the integral of $f(x, y)$ over the projection of the region into the xy plane. This is such an **easy** step that it is **always** better to write volumes as triple integrals to allow other ways to evaluate the integral.

Integrating x over the region gives the **first moment** with respect to the yz plane. The value of this moment can be written $V\bar{x}$, the product of the volume with the x coordinate of the **centroid**. Other first moments are similar.

More Special integrands

Integrating the square of one coordinate, e.g. x^2, or the product of two coordinates, e.g xz, gives **second moments**.

Although one could choose to integrate **anything**, these examples are important enough to be **remembered** for many figures. After computing them once, the value becomes available to use wherever it appears. When these special values are combined with the general **linearity** of the integral, fairly general expressions can be integrated easily.