

## Cylindrical coordinates

If the rectangular coordinates are replaced by polar coordinates in the  $xy$  plane , the result is called **cylindrical coordinates**. Thus, the equations used are

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

## Figures of rotation

Cylindrical coordinates give especially good descriptions of figures having rotational symmetry about the  $z$  axis. Such objects have equations that are independent of  $\theta$ , and properties of the region can be found by drawing a picture in a plane with rectangular coordinates  $r$  and  $z$ . In many cases, it is possible to restrict attention to  $r > 0$  since we allow  $\theta$  to take all values from 0 to  $2\pi$ , allowing all **rays** from the origin in the  $xy$ -plane to be described.

## Spherical coordinates

Replacing these rectangular coordinates by polar coordinates  $\rho$  and  $\phi$  with the positive  $z$  axis as initial direction gives spherical coordinates. The equations relating these systems are

$$z = \rho \cos \phi$$

$$r = \rho \sin \phi$$

$$\theta = \theta$$

## Spherical coordinates, part 2

Since we expect to consider only positive  $r$ , it is necessary only to consider  $0 \leq \phi \leq \pi$  and  $\rho \geq 0$ .

The textbook also gives formulas for the direct conversion between rectangular and spherical coordinates, but I see no reason to try to remember these formulas. They are rather cumbersome and may actually take more effort than using cylindrical coordinates as an intermediate step. It is curious that such formulas have achieved the status of **sliced bread**, when one of the main themes of calculus is the expression of operations as a composition of small steps.

## Exercises 12.7

#55. Express  $x^2 + y^2 = 2y$  in other coordinate systems.

#57. Describe  $r^2 \leq z^2 \leq 2 - r^2$ .

#61. Describe  $-\pi/2 \leq \theta \leq \pi/2, 0 \leq \phi \leq \pi/6, 0 \leq \rho \leq \sec \phi$ .

## Integrals in these systems

The formulas needed to evaluate integrals are exactly those arising in polar coordinates:  $dx dy = r dr d\theta$  becomes

$$dz dx dy = r dz dr d\theta$$

and  $dz dr = \rho d\rho d\phi$  becomes

$$dz dr d\theta = \rho d\rho d\phi d\theta.$$

## Integrals in spherical coordinates

Combining these parts gives

$$dz dx dy = \rho^2 \sin \phi d\rho d\phi d\theta.$$

This shows that it is possible to give formulas connecting rectangular and spherical coordinates, but it also shows that those formulas are direct consequences of the **much simpler** formula for relating rectangular and polar coordinates in a plane.

## Exercises 15.8

In earlier exercises, the textbook tells you the system to use, which may be different from the system used to describe the region. Here, we will investigate all of the options. Also, in general, we will use only verbal descriptions since the notation for the integrals is only useful after you have identified the iterated integrals that you will be computing. Integration is always with respect to **volume**.

#7. Integrate  $\sqrt{x^2 + y^2}$  over the region inside  $x^2 + y^2 = 16$  and between  $z = -5$  and  $z = 4$ .

## More Exercises 15.8

#11. Integrate  $x^2$  over the region inside  $x^2 + y^2 = 1$ , above  $z = 0$  and below  $z^2 = 4x^2 + 4y^2$ .

#17. Integrate  $x^2 + y^2 + z^2$  over the ball  $x^2 + y^2 + z^2 \leq 1$ .

#23. Find the volume of the solid that lies above the cone  $\phi = \pi/3$  and below the sphere  $\rho = 4 \cos \phi$ .