The Divergence Theorem, part 1

The proof of the divergence theorem is a three dimensional version of the proof of Green's theorem. It is proved first for regions for which all sections parallel to one of the coordinate axes are intervals (or empty). Evaluating a triple integral of an expression f over a solid body $\mathcal{B}$ as an iterated integral begins by using the fundamental theorem of one-variable calculus to find an expression R such that $\partial R/\partial z = f$. The **original triple integral** is then evaluated as the **double integral over the projection** in the xy plane of the difference of R for z at the top and bottom of $\mathcal{B}$.

The Divergence Theorem, part 2

However, the operation of substituting the value of z on a surface and integrating R with respect to x and y is exactly the same as the surface integral of $R\mathbf{k}$ over the surface.

Any part of the surface perpendicular to the xy plane gives a zero contribution to such a surface integral. The **outward normal** to the boundary $\mathcal{S}$ of $\mathcal{B}$, **normalized** to give an integral with respect to area in the xy plane has third component $+1$ on the top and third component -1 on the bottom. Since we have chosen a vector field whose first two components are zero, it is only the third component of the normal to the surface that contributes to the integral.

The Divergence Theorem, part 3

In the same way, on can integrate first with respect to x or y. This gives the three parts of the divergence theorem:

$$\iint_{\mathcal{S}} \langle P, 0, 0\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial P}{\partial x}\, dV$$

$$\iint_{\mathcal{S}} \langle 0, Q, 0\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial Q}{\partial y}\, dV$$

$$\iint_{\mathcal{S}} \langle 0, 0, R\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial R}{\partial z}\, dV$$

The Divergence Theorem, part 4

Adding these shows that the the integral of a vector field $\mathbf{F} = \left\langle P, Q, R \right\rangle$ over $\mathcal{S}$ is equal to the integral of the **scalar quantity**

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{$*$}$$

over $\mathcal{B}$.

The quantity $(*)$ is called the **divergence** of $\mathbf{F}$ and denoted $\nabla \cdot \mathbf{F}$. You should **see** how this notation is a mnemonic for $(*)$.

The general form of the divergence theorem

The divergence theorem may be extended to more general regions that can by cut into pieces having the form used in the proof. Since **the surface integral is oriented**, its value on a cut will be counted with opposite signs when considered as part of the boundary of regions on the two sides of the cut. When one adds over the dissection of the region, the contribution of each cut simplifies to zero.

The general form of the divergence theorem, part 2

For simple regions, the evaluation of the surface integral appearing in the divergence theorem gives **exactly** the integral over the projection into one of the coordinate planes resulting from evaluating the innermost integral in the description of the triple integral as an iterated integral.

This knowledge can be used to improve understanding of surface integrals.

How is the divergence theorem used? (part 1)

Most textbook exercises in the use of the divergence theorem use the triple integral to evaluate the surface integral. Since this is opposite to the direction of the proof, some explanation should be given. There are several reasons for this.

(1) It is certainly true, as in the proof of the divergence theorem, that the first step in the evaluation of a triple iterated integral over $\mathcal{B}$ can be interpreted as a surface integral over the boundary of $\mathcal{B}$. However, what the calculation **actually gives** is a double integral over the projection of $\mathcal{B}$ into one of the coordinate planes, which is a simple interpretation of the surface integral that **appears to** bypass the general notation for surface integrals.

How is the divergence theorem used? (part 2)

(2) The different components of the surface integral appear in the proof of the divergence theorem as the results of using different approaches to setting up an integral over $\mathcal{B}$ as in iterated integral. Normally, one only writes the integral in the one form that will be **easiest to evaluate**. When different terms of the triple integral are evaluated using different orders of integration, you are essentially choosing to assign those terms to different parts of the divergence. However, **theses operations are independent**. The method used to evaluate the surface integral need not be dependent on the way that the integrand is constructed.

How is the divergence theorem used? (part 3)

(3) When the integrand in the surface integral is expressed in terms of polynomials, it becomes simpler when differentiated. Thus the integrand of the triple integral appears simpler than that of the surface integral.

(4) Certain triple integrals are known because they represent volumes or moments that are remembered from previous computations. It is not necessary to repeat these computations in order to get the answer, and the coordinate system of the current problem may make those computations tedious. **Since the goal of these calculations is to get correct answers, some effort should be made to formulate problems in a way that simplifies computation.**

Exercises 16.9 #3, part 1

$\mathbf{F} = \langle 3x, xy, 2xz \rangle$. Region is cube in which each coordinate is between 0 and 1.

The cube has 6 faces. Integration over each face looks very different from integration over a different face. Since the normal vectors to the faces are in the coordinate directions, $\mathbf{F} \cdot \mathbf{n}\, dS$ will extract **single components** of $\mathbf{F}$ on each face.

Exercises 16.9 #3, part 2

We treat the faces one at a time.

$x = 1$: $\mathbf{n} = \mathbf{i}$, $\mathbf{F} \cdot \mathbf{n} = 3x$, but **this is to be evaluated** at $x = 1$ giving the constant function 3. Integrating over the **unit square** in the yz plane gives a value of 3.

$x = 0$: $\mathbf{n} = -i$, $\mathbf{F} \cdot \mathbf{n} = -3x$, but **this is to be evaluated** at $x = 0$ giving the constant function 0. Integrating over the **unit square** in the yz plane gives a value of 0.

Exercises 16.9 #3, part 3

$y = 1$: $\mathbf{n} = \mathbf{j}$, $\mathbf{F} \cdot \mathbf{n} = xy$, but **this is to be evaluated** at $y = 1$ giving x. Integrating over the **unit square** in the xz plane gives a value of $1/2$.

$y = 0$: $\mathbf{n} = -j$, $\mathbf{F} \cdot \mathbf{n} = -xy$, but **this is to be evaluated** at $y = 0$ giving the constant function 0. Integrating over the **unit square** in the xz plane gives a value of 0.

Exercises 16.9 #3, part 4

$z = 1$: $\mathbf{n} = \mathbf{k}$, $\mathbf{F} \cdot \mathbf{n} = 2xz$, but **this is to be evaluated** at $z = 1$ giving $2x$. Integrating over the **unit square** in the xy plane gives a value of 1.

$z = 0$: $\mathbf{n} = -k$, $\mathbf{F} \cdot \mathbf{n} = -2xz$, but **this is to be evaluated** at $z = 0$ giving the constant function 0. Integrating over the **unit square** in the xy plane gives a value of 0.

The sum of the face integrals is

$$3 + 0 + \frac{1}{2} + 0 + 1 + 0 = \frac{9}{2}$$

Exercises 16.9 #3, part 5

The divergence of **F** is

$$\frac{\partial}{\partial x}(3x)+\frac{\partial}{\partial y}(xy)+\frac{\partial}{\partial z}(2xz)=3+x+2x=3+3x.$$

Integrating first with respect to x gives

$$\int_0^1 3+3x\,dx=3+\frac{3}{2}=\frac{9}{2}$$

Integrating this over a square of area 1 again gives $9/2$.

Exercises 16.9 #5, part 1

$\mathbf{F} = \langle xy, yz, zx \rangle$. Region is solid cylinder defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$.

The boundary consists of two circular faces in the planes $z = 0$ and $z = 1$ and a cylindrical band. In planes parallel to the xy plane, we will integrate with respect to x and y — or perhaps convert to polar coordinates r and θ. The cylinder should be parameterized by z and θ with $x = \cos\theta$ and $y = \sin\theta$.

Exercises 16.9 #5, part 2

$z = 1$: $\mathbf{n} = \mathbf{k}$, $\mathbf{F} \cdot \mathbf{n} = zx$, but **this is to be evaluated** at $z = 1$ giving x. Integrating over the circle $x^2 + y^2 \leq 1$ gives 0, since the centroid of this circle is at the origin..

$z = 0$: $\mathbf{n} = -k$, $\mathbf{F} \cdot \mathbf{n} = -zx$, but **this is to be evaluated** at $z = 0$ giving 0. Integrating gives 0.

Exercises 16.9 #5, part 3

Our parameterization of the cylinder requires

$$\begin{aligned} \mathbf{n}\,dS &= \frac{d\mathbf{r}}{d\theta} \times \frac{d\mathbf{r}}{dz}\,d\theta\,dz \\ &= \big\langle -\sin\theta, \cos\theta, 0 \big\rangle \times \big\langle 0, 0, 1 \big\rangle\,d\theta\,dz \\ &= \big\langle \cos\theta, \sin\theta, 0 \big\rangle\,d\theta\,dz \end{aligned}$$

(where some pre-processing was involved to assure that the **outward** normal was obtained). Note that this parameterization expresses the surface of the cylinder as a rolled-up θz-plane.

Exercises 16.9 #5, part 4

On the cylinder,

$$\mathbf{F} = \left\langle \cos\theta \sin\theta, z\sin\theta, z\cos\theta \right\rangle$$

$$\mathbf{F}\cdot\mathbf{n}\,dS = \cos^2\theta\sin\theta + z\sin^2\theta\,d\theta\,dz$$

To cover the desired surface **exactly once**, requires $0 \le \theta \le 2\pi$ and $0 \le z \le 1$. Now,

$$\int_0^{2\pi} \cos^2\theta\sin\theta + z\sin^2\theta\,d\theta = \pi z$$

$$\int_0^1 \pi z\,dz = \pi/2$$

Exercises 16.9 #5, part 5

The divergence of **F** is

$$\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x.$$

The integrals of x, y and z are moments and symmetry tells us the location of the centroid to be $\langle 0, 0, 1/2 \rangle$. We also know the volume to be π, so the integral is $\pi/2$. Direct evaluation of these integrals is not difficult.

Further properties of divergence

We noted that Stokes’ theorem told us that the flux integral of the curl of a vector field over a closed surface is always zero. The divergence theorem then suggests that this must be due to the divergence of this vector field being zero. It is easy to verify this directly using the equality of mixed partial derivatives.

Further properties of divergence, part 2

Green's theorem was originally obtained as the **leading special case** of Stokes' theorem relating a **flow** integral along a curve to a double integral over the region bounded by the curve. We did this because that was the **only kind of line integral that we knew**. However, both the vector field and the tangent vector can be rotated through a right angle to interpret this integral as a **flux** integral. This gives Green's theorem as a two dimensional form of the divergence theorem.