Figures of rotation

If a solid is formed by rotating a plane figure about the z-axis, then we take

$$dV = dA\,dz$$

allowing the integral to be considered as the integral with respect to z of the integrals over the **sections** parallel to the xy-plane.

These sections are typically **disks** or **washers**, and always easily described in terms of circles centered at the origin.

Figures of rotation, part 2

It makes sense to use **polar coordinates** to evaluate these integrals, so that

$$dA = r\, dr\, d\theta$$

Rearranging the order of integration, we have

$$dV = r\, dr\, dz\, d\theta$$

Note the presence of the factor r that is the **Jacobian** of the change of coordinates **in space as well as in the plane**.

The axial plane

This allows the figure being rotates to be described in an rz-plane, which we call the **axial plane**. The symbol r represents **horizontal distance independent of direction**. A picture of the domain of integration can be presented more simply by giving a **two-dimensional** plot in this plane instead of a **three-dimensional** plot that aims to convey the appearance of the solid body, but **contributes nothing to calculating the integral**.

The theorem of Pappus

A classical application of this approach is a Theorem appearing in Section 8.3 of the textbook (page 568).

Theorem of Pappus. *If a plane region is rotated about a line not meeting the region, the volume of the resulting solid is the product of the area of region and the distance traveled by the centroid.*

The theorem of Pappus, part 2

Equivalently, the volume is 2π times the **first moment** of the region with respect to the line since that moment is the area times the product of the distance to the centroid.

The proof (on the next slide) shows that this is a general feature of **all** integrals over figures of rotation.

Proof of the theorem of Pappus

Volume is just the integral of the constant 1 with respect to $dV = r\,dr\,dz\,d\theta$. The integral with respect to r and z is the moment, **independent of θ**, so the last step if the integration of **this constant** with respect to θ from 0 to 2π. The limits of integration on θ are forced by the requirement that **the domain of integration should cover the intended figure exactly once**. Descriptions in rectangular coordinates usually have this property automatically, but it is a key consideration when other coordinate systems are used.

Spherical coordinates

Once attention has been focused on an integral in the axial plane, any method for evaluating that integral can be used. In particular, a **polar coordinate system** in that plane can be constructed. Traditionally, these coordinates are called ρ and ϕ and the **initial direction** $\phi = 0$ is taken to be the positive z-axis. Since figures in the axial plane usually lie on one side of the axis of rotation, it is customary to require $\rho \geq 0$ and $0 \leq \phi \leq \pi/2$.

Spherical coordinates, part 2

The equations for this change of variable are

$$r = \rho \sin \phi$$

$$z = \rho \cos \phi$$

Any r and z in the integrand — including the r coming from the Jacobian — are converted by these equations. There is the additional equation

$$dr\, dz = \rho\, d\rho\, d\phi$$

Since the use of polar coordinates is so familiar, and since the axial plane is often useful in interpreting integrals, these two easy steps may be more useful than a single formula giving a direct conversion between rectangular and spherical coordinates.