Mathematics 251 Maple Lab 1

Vector Calculus

Spring 2006

This project The worksheet in the **seed file** contains instructions to guide you through one example, with space left in other **Sections** for you to fill in the modified instructions needed for the other examples. Space is left at the top of the worksheet for you to use the **text** feature of Maple to add a header containing your name and any other information requested by your instructor. This worksheet should be printed and submitted for grading.

The seed file begins with `with(plots):with(VectorCalculus):` to load the libraries we need.

Correct interpretation of formulas in this project depends on being able to multiply a **vector function** and a **scalar function** and get a **vector function** in which each component of the vector factor has been multiplied by the scalar factor. Unfortunately, the `dot` operator gives a **formal product** that isn't recognized as a vector unless the scalar factor is constant. At the present time, there is an **undocumented feature** that does what we want: the `star` operator for ordinary multiplication or `slash`for division give the desired result. This simpler approach will be used in this project description, and an example will appear in the seed file. Since this is contrary to Maple documentation, it must be tested before relying on it to get correct answers. The supplementary worksheet includes such a test as well as the following more cumbersome construction that agrees with the documentation.

```
scalMult:=(s,V)->LinearAlgebra:-ScalarMultiply(V,s);
```

The `scalMult` defined here has the scalar as its first argument to agree with the convention of the textbook, while the library function expects its arguments in the reverse order. This change is easy to make when defining a function.

Finally, there is the instruction `SetCoordinates( cartesian[x,y,z] );` to assure that all objects are interpreted in rectangular coordinates. Maple allows expressions to be defined in many different coordinate systems, but the rectangular (or `cartesian` coordinates are easier to interpret, so they will be used exclusively in this project.

A **supplementary worksheet** is also available from the web page that suggests experiments to be done on a separate worksheet. Although this supplementary worksheet should not be submitted for grading, it provides a place to test Maple commands and save examples that you would need to edit out if done on the main worksheet. Work done on this sheet should be submitted for grading; it only serves as a place for explorations of related topics in Maple. In particular, an undocumented alternative to `scalMult` was added to the Spring 2006 supplementary worksheet.

In order to refer to results computed earlier in a worksheet, it is common to **assign** those results to a name. An alternative, introduced in Maple 10, is an **equation label** used to identify the last result in an **execution group**. When a result is needed, its equation number may be inserted through the menu bar of your Maple window, a context menu, or a keyboard shortcut (typically CTRL-L). The effect is to bring up a dialog box in which you enter the current label. These labels are automatically updated if commands are inserted or deleted.

1. The helix

Consider the curve

$$\mathbf{r}(t) = \langle \cos(12t)\,,\ \sin(12t)\,,\ 5t \rangle. \qquad (H)$$

The seed file contains instructions for constructing a `Vector` named `Helix1` with these components and obtaining two views of a graph of the curve. (The instructions include several options that turn out to be necessary; the supplement shows the difficulty with attempts to use simpler instructions.) To assure that the graph in your worksheet always has the view you want, an `orientation` option is included. The numbers in this option are the **spherical coordinates** (in degrees) of a direction in space. If you **drag** an existing plot, these numbers appear as θ and ϕ on the **context bar** near the top of the screen. When you get a suitable view, record these values (either by hand or by copying them to the input line of a different worksheet) and enter them in the `orientation` option of the `spacecurve` command that you are constructing. Note that the **side view** shows that different values of the parameter t give different points on the curve., but the **top view** shows a projection that intersects itself several times.

Discussion

Looking straight down the axis of this helix ($\phi = 0$ and any θ) gives the projection into the xy plane. **What is this projection**? **Describe** how the given parameterization $\mathbf{r}(t)$ is used to identify this projection. (You are encouraged to also experiment with a moving a graph drawn by Maple into this orientation, but your discussion should concentrate on the role of the equation.)

Arc length and Curvature

The derivative of `Helix1` is a vector (called `TdsH` in the seed file) whose length is ds/dt and whose direction is the **tangent vector** to the helix. A general method of finding the length of a vector is to take the **square root** of the **dot product** of the vector with itself. This is done for `TdsH` in the seed file. You should see that Maple's value for this expression **could be simpler**, so the `simplify` instruction is applied to this result. The ordinary symbol for division (`/`) is used to divide a vector by its length. If this undocumented feature doesn't work, follow the hints in the supplementary worksheet for using the `scalMult` function. This gives the **unit tangent vector** — denoted `TH`.

One way to find curvature is to differentiate the **unit tangent vector** with respect to s to obtain a vector whose length is the curvature and whose direction is the **unit principal normal**. Although all vectors are known only as functions of t, the chain rule allows us to find the derivatives we want (again as functions of t) by dividing by the expression for ds/dt found above. Maple commands for doing this, similar to those used to find ds/dt and the unit tangent vector from the vector parameterizing the curve are included in the seed file.

Direct application of this description (as well as similar formulas in other parts of this problem) is likely to lead to a messy expression. In order to compare the results of the three computations of curvature, you must `simplify` the answers, although you may need to force some transformation before Maple knows what you consider to be simple. In all cases, you should be able to obtain an **understandable** formula for the curvature. If Maple includes symbols that you don't understand, you have probably not described the formula correctly. In particular, you must be careful to distinguish vector and scalar quantities and use the appropriate operation for the object you have. A variable in Maple may represent any kind of object, and the effect of an operation will depend on the type of object involved when the command is executed.

Supplements

The supplementary worksheet explores functions in the VectorCalculus package that allow features of the curve to be summoned by name instead of by using the recipe described in the textbook. The longer process is used in the main part of this project to have easier access to intermediate results and to study the **method** of computing these objects.

2. Another curve

Consider the curve

$$\mathbf{r}(t) = \left\langle 2 - 3(\cos^2 t - \sin^2 t)\,,\ \cos t\,,\ \sin t \right\rangle. \qquad (Q)$$

Construct a `Vector` with these components, and **assign** it the name `Q`. Note that $(\cos^2 t - \sin^2 t)$ is written in Maple as `(cos(t)^2-sin(t)^2)`.

Three methods for finding curvature will be described here. They are closely related, but the details of the computation are different. In the remainder of this project, we will investigate how the differences in computation affect the way the answer is presented. In describing these computations, we use $\mathbf{r}(t)$ for the **vector function** parameterizing the curve. If this represents the way that the curve is drawn with t representing time, its derivative with respect to t, $\mathbf{r}'(t)$ can be thought of as the **velocity vector** $\mathbf{v}(t)$, and $r''(t)$ is the **acceleration vector** $\mathbf{a}(t)$.

In the **geometric** approach, one pretends that the curve is parameterized in terms of **arc length** s. A description of s as a function of t is developed early in this study. Although this function is frequently difficult to describe, its derivative ds/dt is easily found — it is just the length of $\mathbf{v}(t)$. The chain rule tells us that we can get differentiate with respect to s by differentiating with respect to t and dividing by ds/dt. In this way, all the quantities that we need are given in terms of t, even if we are following an intrinsic geometric procedure based on a parameterization by arc length.

Graphing

Begin by **plotting** this curve using the `spacecurve` command as in the previous example. Experiment with the plot and select an `orientation` that you think best represents the shape of the curve. Be sure to include a `title` option and the values of θ and ϕ for your `orientation` option.

Curvature formula 1

The **velocity vector** $\mathbf{v}(t) = \mathbf{r}'(t)$ can be written as the product of its **length** and a **unit vector** giving its **direction**. The process (and the Maple commands) for finding the length of a vector and using it to scale the vector to obtain its direction was shown in the previous example. The length of $\mathbf{v}(t)$ is **by definition** ds/dt and its **direction** is the **unit tangent vector** $\mathbf{T}(t)$. Note that one always **finds the length first** and uses the `scalMult` function defined at the start of the seed file to **scale the vector** by the factor needed to obtain a **unit vector**.

Now, $d\mathbf{T}/ds$ is a vector whose length is the **curvature** κ and whose direction is the **principal normal** $\mathbf{N}$. Obtain this vector by differentiating $\mathbf{T}$ with respect to t and scaling the result by the reciprocal of ds/dt. Once you have this vector, find its length. This is κ.

Curvature formula 2

The formula

$$\kappa_1 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{v}$ is the **velocity vector** and $\mathbf{a} = d\mathbf{v}/dt$ is the **acceleration vector**, gives a different computation of the curvature. Here, the **absolute value notation** is used to denote the **length** of a vector. This change in the **mathematical notation** should not affect how you use Maple to find the length of a vector.

This formula has the advantage that it is expressed entirely in terms of $\mathbf{r}(t)$ and its derivatives **with respect to the original independent variable** t. It is also the easiest computation to **express as a single formula**, although that **may not be the best way to compute it**. Here, $|\mathbf{v}|$ is the quantity identified as ds/dt in the first formula. Look up `CrossProduct` in the `VectorCalculus` help pages for information about using `&x` to obtain the cross product. (The **supplementary worksheet** also contains an example of this notation for the cross product using vectors from the helix example.)

Curvature formula 3

Another method of computing κ was suggested in Section 13.4, where the formula

$$\mathbf{r}''(t) = \frac{d^2 s}{dt^2}\mathbf{T} + \kappa \left(\frac{ds}{dt}\right)^2 \mathbf{N}$$

was obtained. This is an expression for $\mathbf{r}''$ in terms of the perpendicular unit vectors $\mathbf{T}$ and $\mathbf{N}$.

You have already given names to the **velocity vector** $\mathbf{v}$ and the **acceleration vector** $\mathbf{a}$. The **vector projection** of $\mathbf{a}$ on the direction of $\mathbf{v}$ is given by

$$\operatorname{proj}_{\mathbf{v}} \mathbf{a} = \frac{\mathbf{v} \cdot \mathbf{a}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$$

and $\mathbf{a} - \operatorname{proj}_{\mathbf{v}} \mathbf{a}$ is perpendicular to $\mathbf{v}$. Since $\mathbf{v}$ has the direction of $\mathbf{T}$, this is the component in the direction of $\mathbf{T}$. Hence, $\mathbf{a} - \operatorname{proj}_{\mathbf{v}} \mathbf{a}$ is the component in the direction of $\mathbf{N}$. Thus, dividing the length of this vector by $\left(ds/dt\right)^2$ gives κ. Although not as easy to express as a formula, this method has the advantage of not requiring the cross product.

Conclusion

You have found three expressions that all claim to be κ. They really **must** give the same answer. **Describe** any differences between the **simplest forms found by Maple** from the three computations, together with **your evidence** that they are really the same.

End of Lab 1