Mathematics 251 Maple Lab 4

Triple Integrals

Spring 2006

This project The worksheet in the **seed file** is divided into **Sections** corresponding to the parts of this project description. It also contains commands to load libraries and make common definitions. Your main worksheet will grow from this file. You should remove any extraneous material from the worksheet, but you may elaborate on the organization in preparing your report. There is also a **supplementary worksheet** that contains instructions to refine the Maple commands used in the main worksheet. The instructions for producing graphs are to be copied to the main worksheet after using the supplementary worksheet to experiment with the plot to obtain suitable values for all options. Please turn in only the printout of your **main Maple worksheet**. Use the **text** feature of Maple to add a header containing your name and any other information requested by your instructor. Use the **title** option in all plots to introduce a label that will be kept with the plot when your report is printed.

In this lab, we use Maple to help visualize and compute the value of a triple integral. In particular, we see that the difficulty of the calculation can vary with the **presentation of the integral**, although all **correct formulations** of the same integral will lead to the same answer (barring errors in computation).

The basic command for integration in Maple is `int`. To evaluate $\int_0^1 (x+1)\,dx$, we type

```
int(x + 1, x = 0..1);
```

To evaluate the double integral $\int_0^1 \int_0^{1-x} (xy + x + 1)\,dy\,dx$, we type

```
int(int(x*y + x + 1, y = 0..(1-x) ),x = 0..1);
```

or mimic the way you evaluate such integrals by hand by the two steps

```
Iy:=int(x*y + x +1, y = 0..(1-x) );
int(Iy,x = 0..1);
```

To evaluate the triple integral $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (xyz + x + 1)\,dz\,dy\,dx$, we type

```
int(int(int(x*y*z + x + 1, z = 0..(1-x-y) ),y = 0..(1-x) ),x=0..1);
```

or explicitly perform the steps of the iterated integrals, naming the intermediate steps to allow them to be used in later steps. Details are shown in the **supplementary worksheet**.

1. Domain of integration We will be calculating an integral over a solid S defined to be the region between

$$y = 12 - z^2 - 3x^2 \qquad (1) \qquad \text{a paraboloid}$$

and

$$y = 2z^2 \qquad (2) \qquad \text{a parabolic cylinder}$$

(a) Since the equation of each boundary of S is given by an equation expressing y as a function of x and z, it will be easiest to calculate integrals over S by doing the y integration first. This requires identifying an interval of values of y, depending on x and z, that expresses the phrase "between the surfaces". The first step will be to interpret this graphically.

The basic commands for defining the surfaces and a rectangle containing the domain are

```
y1:=12-z^2-3*x^2; y2:=2*z^2;
Rect:=x=-2..2,z=-sqrt(12)..sqrt(12);
```

These are given in both the seed file and the supplementary worksheet. A `plot3d` command is given in the supplementary worksheet to get a view of these two surfaces on the same set of axes, including axes. The surfaces are shown in **solid colors** to emphasize the **curve** where the two surfaces intersect (although the shape may be distorted because of the `scaling=UNCONSTRAINED` option). Experiment with different views of this figure and **choose a view** that shows one surface marking the lower bound of y and the other marking the upper bound of y for the solid S. Record the values of θ and ϕ and put them in an `orientation` option. Add this `orientation` option and a `title` option, and copy the complete `plot3d` command to your main worksheet.

(b) Give a **text discussion** of:

($*$) which function is the lower bound on y for points of S?

($*$) which function is the upper bound on y for points of S?

(c) Now, we need to determine the x and z limits of integration. To get these, it may be helpful to view the **projection of the intersection** of the paraboloid and the parabolic cylinder onto the x-z plane. To get this projection, use the command `implicitplot(y1=y2,Rect,scaling=CONSTRAINED);` (the `scaling=CONSTRAINED` option assures that you will see the **shape** of the curve). This command is in the seed file, but you will need to **add a title**.

When you have the graph add the following **text discussion**:

($*$) What type of curve is this projection of the intersection of the two surfaces?

($*$) Is the intersection itself the same type of curve?

($*$) Give an equation of the projection of the intersection.

(d) The **interior** of the curve found in part (c) is the projection of S into the xz-plane. Use this to find bounds on z depending on x that identify points in this region; then give constant bounds on x that identify where these bounds on z are valid (the bounds on x will describe the **projection** of S on the x-axis). Use a **text statement** to

($*$) Describe the bounds on this x and z.

Then, **test your bounds** by repeating the `plot3d` command of 1a with `Rect` replaced by a Maple version of this description of the projection in the xz plane. The result should be resemble a solid body (limitations of computation may leave a small hole in the surface). The **supplementary worksheet** illustrates this for the region bounded by the paraboloid (1) and the xz plane. Use the `style=PATCHNOGRID` option along with an `orientation` option chosen after positioning your first plot and a `title` option that is required for all plots in the main worksheet.

2. The first integral

If a solid has mass density $\delta = 1$, then its moment of inertia about the z-axis is given by

$$I_z = \int\int\int (x^2 + y^2)\, dV.$$

We wish to calculate I_z for a solid S defined in part 1.

Formulate and evaluate the triple integral I_z using three applications of the Maple command `int` with the limits of integration found in part 1. You may write the integral either as a single statement or use an explicit iteration as described at the start of this project description (and illustrated in the supplementary worksheet).

Note: If you were doing this by hand, you would probably choose another way to compute the integral with respect to x and z, but it is better to use the most direct description when Maple is doing the computation.

The **supplementary worksheet** illustrates the **vector calculus** approach to this integral. Using the **proof** of the **divergence theorem**, the first integration in a triple integration may be interpreted as constructing a **vector field** whose **flux integral** on the boundary is an expression for the next two integrals to be performed in evaluating the triple integral. Although we won't explore the possibility, this approach allows alternate descriptions of the boundary to be introduced while preserving the first integral.

3. Alternative description of domain

(a) To do the z integration first, solve equations (1) and (2) for z. Maple can do this, but you may find it easier to solve the equations by hand. When you have the solutions, assign the names `z1` and `z2` to the Maple expressions for the positive and negative solutions, respectively, of equation (1) and the names `z3` and `z4` to the Maple expressions for the positive and negative solutions, respectively, of equation (2). These four equations each specify a portion of the boundary of S. Plot these four surfaces on a single set of axes over the region $-2 \le x \le 2$, $0 \le y \le 12$.

As usual, you should experiment with different views of this figure in the supplementary worksheet in order to choose an `orientation` option that you will use, along with a `title`, in the plot command in the main worksheet.

(b) You must now determine the limits of integration in x and y, which is the projection of S into the xy-plane. The set S_0 consists of all points (x, y) for which **all four functions** graphed in (a) are defined. You should see that this is a region between the line $y = 0$ and a curve that has an equation of the form $y = f(x)$. Determine (by hand) $f(x)$ and assign the name `f` to the Maple expression for $f(x)$.

(c) Note that the plot in (a) shows that for some values of x and y the z values inside S range from the bottom of the paraboloid to the top of the paraboloid, while for other values of x and y, the z values inside S range from the bottom of the parabolic cylinder to the top of the parabolic cylinder. Thus, to evaluate I_z we need to write it as the sum of two integrals I_1 and I_2 depending on which surfaces form the boundary. These two regions are separated by a curve of the form $y = g(x)$ that is the projection on the xy plane of the intersection of the surfaces $z1$ and $z3$. Determine an equation for this curve in the form $y = g(x)$ and assign the name `g` to the Maple expression for $g(x)$.

(d) To see what the two regions look like, plot the curves $y = 0$, $y = f(x)$, and $y = g(x)$ for $-2 \le x \le 2$ on the same set of axes. You should use the **two-dimensional** `plot` command with the `axes=BOXED` option to show the relation to the coordinates.

4. The second integral

(a) Let I_1 be the integral over the portion of S that projects into the region between $y = 0$ and $y = g(x)$. Identify the limits of integration on z in integral and write the Maple expression for this triple integral, giving it the name `I1`.

(b) Let I_2 be the integral over the portion of S that projects into the region between $y = g(x)$ and $y = f(x)$. Identify the limits of integration on z in integral and write the Maple expression for this triple integral, giving it the name `I2`.

(c) Now ask Maple to find the sum of I_1 and I_2. If the answer is not the same as the result of part 2, **there is a mistake somewhere** in your worksheet. Please correct it before submitting your report. Then, conclude your report by declaring that you have found I_z by two different computations.

The supplementary worksheet continues the investigation of the **vector calculus** approach to this integral. Again, the **proof** of the **divergence theorem** says that the first integration in a triple integration may be interpreted as constructing a **vector field** whose **flux integral** on the boundary is an expression for the next two integrals to be performed in evaluating the triple integral. Integration with respect to z is used to find this vector field. However, the **simpler description** of the surface used in part 2 can be used in place of the complicated description used when the boundary must be described in terms of x and y. Only the substitute integral is shown in the worksheet, but it should be easy to adapt the description to any integral.

End 251 lab 4 (Fall 2004)