Math 251:07–09 — Spring 2001
TF3 PH-111
Prof. Bumby
Lab 1 VECTOR CALCULUS

This description of the lab is only for the sections identified in the header. Other sections will use a slightly different version. A *seed file* `lab1.mws` for this version can be downloaded from the web page for this section. That file will assure that the descriptions given here are accurately transcribed to your worksheet.

Please turn in only the printout of your Maple worksheet. Use the **text** feature of Maple to add a header containing your name, identify the separate parts of the lab, labels for the plots in your worksheet and explicit answers to all questions asked (do not write this material in by hand). Remove from the worksheet any extraneous material and any errors you have made. To be sure that the input and output shown agree, you should use a `restart` command and then select **Execute Worksheet** from the Edit Menu.

Introduction. In addition to the `plots` library that you load using the `with(plots)` instruction, the basic operations with vectors will be needed. The seed file contains the definitions

```
scalP:=(sc,vec)->map(x->sc*x,vec);
dotP:=(v1,v2)->sum(v1[i]*v2[i],i=1..nops(v1));
crossP:=(v1,v2)->[v1[2]*v2[3]-v1[3]*v2[2],
         v1[3]*v2[1]-v1[1]*v2[3],v1[1]*v2[2]-v1[2]*v2[1]];
proJ:=(a,b)->scalP(dotP(a,b)/dotP(a,a),a);
```

These create functions with special names to multiply the scalar that is the first argument of `scalP` by the vector (actually a *Maple list*, which is a simpler structure than is usually used for vectors) that is the second argument; to take the dot product or cross product of the two vectors that are the arguments of `dotP` or `crossP`; or to find the projection of the second argument of `proJ` on the direction of the first. The definitions of these functions should look familiar.

1. Testing the definitions. Introduce vectors **v** and **w**. Find their cross product and resolve **w** into its projection $\mathbf{w}_1$ onto the direction of **v** and $\mathbf{w}_2 = \mathbf{w} - \mathbf{w}_1$, which should be perpendicular to **v**. Here are the instructions.

```
v:= [1,2,3]; w:= [-1,1,2];
cpvw:= crossP(v,w);
w1:=proJ(v,w); w2:=w-w1;
dotP(v,cpvw); dotP(w,cpvw);
dotP(v,w2); # three products that should be zero
lenv:= sqrt(dotP(v,v));  # compute the length of v
Uv := scalP(1/lenv,v); # produces a unit vector in the direction of v
w[1];   # gives the first component of w
```

Note that addition and subtraction behave in the expected way with this representation of vectors. Multiplication and division by scalars *sometimes* give the scalar product, but `scalP` has been defined to always give a result that will act like a vector. Individual components of a vector are named by putting the index in square brackets.

2. Consider the curve

$$\mathbf{r}(t) = \left\langle 1 - 2 * (\cos^2 t - \sin^2 t), \cos t, \sin t \right\rangle.$$

Since all components are expressed in terms of $\sin t$ and $\cos t$, $\mathbf{r}(t + 2\pi) = \mathbf{r}(t)$. Introduce a variable r to represent the vector expression, and use the `spacecurve` instruction with an interval of $-\pi \le t \le \pi$ to get a picture of the whole curve. Experiment with different views of the curve, and choose one that seems to suggest the general appearance of the curve in space.

Also experiment with different intervals of values of t. Do not leave the graphs in the worksheet, but add *text* describing the effect of a slightly interval of values of t and of a slightly larger interval.

3. The following formulas for the unit tangent vector T, normal vector N, and the curvature κ can be found in your book:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}, \qquad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}, \qquad \kappa = \frac{|d\mathbf{T}/dt|}{|\mathbf{v}|}.$$

Use Maple to define expressions for each of these, using the letters given above (note that *Maple* recognizes “kappa” as the name of κ), and then simplify these expressions as much as possible. Note that Maple often defines intermediate quantities (such as `%1, %2`) and then writes the desired expression using these quantities.

Use the functions defined at the top of the worksheet to assure that all vectors are represented as *lists*, which gives the best results in this project. Also note that mathematical notation uses $|\mathbf{v}|$ for what this project’s version of Maple calls `sqrt(dotP(v, v))`.

Also, the formula

$$\kappa_1 = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3},$$

where $\mathbf{a}$ is the acceleration vector $\mathbf{a} = d\mathbf{v}/dt$, gives a different computation of the curvature. We use a different name here since we want to implement this formula in Maple and compare the result to the previously computed value of κ. (You can use the name `kappa1` for the quantity computed by this formula).

Compute simplified forms of both κ and κ_1. Answer the following in *text*: Do you see that they are the same? Does Maple?

4. Another method of computing κ was suggested in Section 13.4. You have already given names to the velocity vector $\mathbf{v}$ and the acceleration vector $\mathbf{a}$. Now $\mathbf{a} - \mathrm{proj}_{\mathbf{v}}\, \mathbf{a}$ is $\kappa(ds/dt)^2\mathbf{N}$. You have the ingredients to compute this using the *proJ* function at the top of the worksheet. This allows κ to be found from the length of this vector and the length of $\mathbf{v}$. Compute a quantity κ_2 that describes the value of κ obtained by this approach. Simplify it and compare to the other values for κ.

Comments. It is a simple matter to construct a tangent line to the curve at a point given by a specific value of t — say $\pi/6$, just to have a concrete example. First, you use `Fig1:=spacecurve(r,t=-Pi..Pi):` to construct (without showing it) a plot structure for the curve. Then `subs(t=Pi/6,r+scalP(u,v))` gives a portion of the tangent line (assuming that the derivative of `r` has been assigned to the variable `v` and `u` has not been defined to be anything but `u`. This can be used to define `Fig2` and then `display(Fig1,Fig2)` will show a graph containing both the curve and a tangent line. Using the calculations in this project, one can produce sketches of the *osculating plane* or *osculating circle* (defined on page 854 of the textbook) of $\mathbf{r}(t)$ at definite points. These sketches give a *visual* demonstration of the correctness of the formulas and their implementation in Maple. However, the construction is not yet concise enough to be described here.