Math 251:07–09 — Spring 2001
TF3 PH-111
Prof. Bumby
Lab 2 QUADRIC SURFACES

This description of the lab is only for the sections identified in the header. Other sections will use a slightly different version. A *seed file* `lab2.mws` for this version can be downloaded from the web page for this section. That file will assure that the descriptions given here are accurately transcribed to your worksheet.

Please turn in only the printout of your Maple worksheet(s). Use the **text** feature of Maple to add a header containing your name, identify the separate parts of the lab, labels for the plots in your worksheet and **explicit answers to all questions asked** (do not write this material in by hand). Remove from the worksheet any extraneous material and any errors you have made. To be sure that the input and output shown agree, you should use a `restart` command and then select **Execute Worksheet** from the Edit Menu.

Previous versions of this project included a large number of graphs that caused *Maple* to crash when they were combined into a single worksheet. The present version has been scaled down, but you should use one worksheet for problems 0 and 1 separate worksheets for each of problems 2 and 3. You should also save your work often. This is good practice in general, but especially important here, in case we have not completely identified the cause of *Maple*'s distress.

The "seed file" contains almost all you need for problems 0 and 1. Imitating the instructions given there should suffice for the other problems.

A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general form of such an equation is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, D, E, F, G, H, I$, and J are constants. In this lab, we use Maple to help visualize some types of quadric surfaces that can arise from equations of the above form.

There are three Maple commands that we will use to plot surfaces in three dimensions. These are `plot3d`, `implicitplot3d`, and `display`. The first is always available, but before the second or third of these can be used, one must type the command `with(plots):`. The command `plot3d` may be used to graph a surface when one of the variables can be solved for explicitly in terms of the other two, e.g., $z = f(x, y)$, while the command `implicitplot3d` is needed to graph any surface of the form $g(x, y, z) = 0$. Any surface of the first form is also of the second form, taking $g(x, y, z) = z - f(x, y)$, so the `implicitplot3d` command is more general. However, the implementation of these instructions is different, so the graphs may not be the same. One part of this project examines the differences between graphs that are the same *in theory*. The command `display` is used to display two previously defined plots at the same time on the same set of axes. Be sure to end a command that creates a named plot for later display with a colon: you really don't want the output it generates. (It won't hurt to see it *once*, but it should not be part of the report you submit.) You may mix plots formed with `plot3d` and `implicitplot3d` in the same `display`, but you may not mix two dimensional plots and three dimensional plots.

The surfaces $z = x^2 + y^2$ and $z = 4$ may also be plotted at the same time by using the syntax

```
plot3d({x^2 +y^2,4}, x=0..1,y=0..1);.
```

However, this limits your options. We are interested in identifying the intersection of two surfaces, so we shall adopt an alternative approach using the `display` command which allows us to choose a different color for each plot. The `setoptions3d` instruction assures that some properties are made part of every plot without special request. Here, we request the boxed form of coordinate axes, equal scales on all axes, and solid colors, since these options are most useful for the questions in the lab. These common features must be executed at the beginning of a session where you work on a later problem.

0. Some *Maple* commands are introduced. You are mostly only required to observe, but part (c) asks for an interpretation of the graphs that were drawn. For these plots, the **title** option of *Maple* plots is used. This gives a way to label plots that is superior to the ordinary text insert requested in previous labs.
Note. One problem with this use of the `display` command is that you do not see any of the plots until the `display` command is used. You can get around this by first typing an instruction like

```
plot3d(1, y=0..1,z=0..2, color=RED);
```

to view the plot. If it is satisfactory, edit the line by inserting `p1:=` at the beginning and changing the semicolon at the end to a colon, and then execute it again. This names the plot and suppresses all output. This instruction appears as a comment in the seed file. If you want to use it, remove the comment symbol at the start of the line. The line should be removed from your final report. Although the final report on this lab should be a concise response to the instructions and questions in this description, you are encouraged to experiment with anything suggested by the work on the report. Your access to *Maple* is not restricted to this course, and we hope that you become sufficiently familiar with it to use it wherever it can be helpful.

(a) Execute the following sequence of commands.

```
with(plots):
setoptions3d(axes=BOXED,scaling=CONSTRAINED,style=PATCH);
p1:=plot3d(1, y=0..1,z=0..2, color=RED):
p2:=plot3d(2, x=0..1,y=0..1, color=BLUE):
display({p1,p2},title="Planes using plot3d");
```

The colors won't appear when printed in black and white, so be sure to identify how different parts were colored when you describe the plots in your answer to part (c). Then,

(b) Execute these commands.

```
p3:=implicitplot3d(x=1, x=0..1, y=0..1,z=0..2, color=RED):
p4:=implicitplot3d(z=2, x=0..1,y=0..1,z=0..2, color=BLUE):
display({p3,p4},title="Planes using implicitplot3d");
```

(c) Insert **text** describing the plots and analyzing why such similar commands give such different graphs.

1. An ellipsoid is a quadric surface of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad (1)$$

where a, b, and c are positive constants. In this problem, we consider the ellipsoid with $a = 2$, $b = 3$, $c = 5$. First, give *Maple* these values. It will then be possible to use general descriptions although these parameters will have specific interpretations when the worksheet is executed. You can use the same worksheet for another example by changing the definitions of a, b, and c at the top of the worksheet and re-executing.

(a) One approach to plotting the surface given by (1) is to solve for z, obtaining the two surfaces

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}, \qquad z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

which correspond to the top and bottom half of the ellipsoid. Each of these surfaces is defined explicitly in the form $z = f(x, y)$ and thus can be graphed using the `plot3d` command. The `plot3d` command requires you to specify a domain over which you wish to plot the graph. For example, `top:=c*sqrt(1-x^2/a^2-y^2/b^2);` is an expression for the top half of the ellipsoid, and `plot3d(top,x=-a..a,y=-b..b);` is a request to plot its graph over a rectangular region containing all points for which the expression `top` can be defined.

It is also possible to construct plots over domains that are not rectangles. This is accomplished by giving bounds on the second variable that depend on the first variable, although it is still required that the bounds on the first variable be constant (at the time that the `plot3d` command is executed. This description of regions in the plane will be useful later in the course, so you should practice it with *Maple*. The exact domain for which the expression `top` yields real values is

$$-a \le x \le a; \qquad -b\sqrt{1 - (x^2/a^2)} \le y \le b\sqrt{1 - (x^2/a^2)}.$$

The “seed file” contains instructions for obtaining three variations on this plot. If the plots are the same size (and not too small, please), they should be identical. However, any difference can become exaggerated when translated into instructions to the computer. End this section with a comment on differences that you observe between the plots.

(b) The top and bottom of the ellipsoid can be plotted together in the same `plot3d` command. The whole ellipsoid can also be plotted using the `implicitplot3d` command. The “seed file” contains instructions for obtaining plots in both styles. Two descriptions of the domain are used with `plot3d`, but the `implicitplot3d` command requires constant bounds. Execute these commands to obtain three plots. Note that the name $E1$ is introduced for one of the plots to allow it to be used in later `display` commands. Add titles to the commands that draw the graphs, and adjust the size of the graphs in your worksheet. Finally, describe any differences between these plots.

(c) Use the instructions in the “seed file” to find the intersection of the ellipsoid $E1$ with the plane $x = 1$. Rotate the plot (and adjust the size) until it clearly shows the curve of intersection. Then, construct a variant to find the intersection of $E1$ with the plane $y = 2$, and a separate plot to show the intersection of $E1$ with the plane $z = 3$.

(d) The intersection of $E1$ with the plane $y = 2$ can easily be found without *Maple*. Do the algebra to find an equation for the intersection, and use this equation to identify the curve. Use the **text** feature of Maple to insert this information. Does this agree with the plots obtained by *Maple*?

Each of the remaining problems should be done on separate worksheets with distinct names. When you switch worksheets within the same *Maple* session, you should execute a `restart` command. Breaking the work into several worksheets also allows independent parts of the project to be done at different times. This level of independence is the best way to avoid difficulties caused by limited machine resources.

2. The surface

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{2}$$

is called an elliptic paraboloid.

Again, set $a = 2$, $b = 3$, $c = 5$, and create plots over the region $-2 \le x \le 2$, $-3 \le y \le 3$.

(a) Solve (2) for z in terms of x and y and use the `plot3d` command. What are the values of z shown in this plot?

(b) Now put the original form of (2) into the `implicitplot3d` command with $-2 \le x \le 2$, $-3 \le y \le 3$ and the range of z found in (a). Is there a difference between these two plots? If so, which do you prefer?

(c) If $x = 1$ and $y = -2$, use your expression for z as a function of x and y from (a) to allow *Maple* to find the value of z such that $(1, -2, z)$ is on the surface. Then continue, allowing *Maple* to do the calculus, to find the direction of the normal vector to the surface at this point. (Find appropriate individual derivatives and assign names — do not try to use the `linalg` package.)

(d) Use the result of (c) to find the equation of the tangent plane at the point and construct a plot of both the surface and its tangent plane.

End this worksheet and begin a new one for the third problem. Use the `restart` command to clear old definitions and make the definitions you need for the problem.

3. The surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{3}$$

is known as an (elliptic) hyperboloid of one sheet. Again, set $a = 2$, $b = 3$, $c = 5$. This time, we see from (3) that the surface can only be defined for (x, y) *outside* the ellipse $x^2/a^2 + y^2/b^2 = 1$, so a larger domain is needed in order to get a good view of the surface, so we choose

$$-2a \le x \le 2a, \quad -2b \le y \le 2b, \quad -2c \le z \le 2c. \tag{3a}$$

(a) Use `implicitplot3d` to obtain a graph of the surface with equation (3) with the bounds given by $(3a)$. As in earlier examples, add a `color` option to the `implicitplot3d` command.

(b) Begin an investigation of the intersection of the surface given by (3) with the plane $z = 2x + y + 2$ by constructing a `plot3d` graph of this equation over the rectangle $-2a \le x \le 2a, -2b \le y \le 2b$ and combine it with the result (a) in a single `display`. Try to find a view that shows the shape of this intersection.

(c) Find the projection of the curve in (b) into the xy-plane by substituting $2x + y + 2$ for z in (3) and constructing an `implicitplot` of the result in a suitable domain in the xy-plane.

End of Lab 2