Math 251:07–09 — Spring 2001
TF3 PH-111
Prof. Bumby
Lab 3 MAXIMUM AND MINIMUM VALUES

This description of the lab is only for the sections identified in the header. Other sections will use a slightly different version. A *seed file* `lab3.mws` for this version can be downloaded from the web page for this section. That file will assure that the descriptions given here are accurately transcribed to your worksheet.

Please turn in only the printout of your Maple worksheet(s). Use the **text** feature of Maple to add a header containing your name, identify the separate parts of the lab, and give **explicit answers to all questions asked** (do not write this material in by hand). Remove from the worksheet any extraneous material and any errors you have made. To be sure that the input and output shown agree, you should use a `restart` command and then select **Execute Worksheet** from the Edit Menu.

In this lab, we use *Maple* to help visualize and compute the maximum and minimum values of a function of two variables. The functions appearing in this lab, like those appearing elsewhere in the course, are usually polynomials, with a few appearances of exponential or trigonometric functions. The names of these standard functions and the usual notation of algebra allow you to write the expressions for their values to which you apply the rules of calculus.

0. As in Lab 2, we begin by loading the `plots` library and fixing some options. Note that, in contrast to earlier work, we replace the `scaling=CONSTRAINED` option with `scaling=UNCONSTRAINED` since it is not necessary to compare distances along different axes (this is the default setting, so it could be omitted — it is included to restore this setting if it had been changed). Then we introduce the expressions

$$\frac{y^3}{9} + 3x^2y + 9x^2 + y^2 + xy + 9 \tag{A}$$

$$y(1-10xy)e^{-x^2-y^2} \tag{B}$$

that will be studied in this project (you should recognize (B) from problem 9 of Lab 0). Although the discussion of this topic in Calculus uses *functions*, it is easier to work with *expressions* in Maple. Throughout this worksheet, x and y will be treated as independent variables. All other names (except for t in 3e) will stand for expressions depending on those variables. That is, they are the values of functions of x and y. Thus, we use the name `A` for the expression in (A). We also introduce the region $\mathcal{R}$ consisting of the lower half (i.e. $y \le 0$) of the elliptical disk $x^2 + y^2/49 \le 1$ that will be used in problem 3. The partial derivatives of A and B are also found here. These derivative are needed often in this project, so they need names, and `Ax` is a convenient abbreviation for $\partial A/\partial x$.

```
with(plots):
setoptions3d(axes=BOXED,scaling=UNCONSTRAINED,style=PATCH);
A:=y^3/9+3*x^2*y+9*x^2+y^2+x*y+9;
B:=y*(1-10*x*y)*exp(-x^2-y^2);
R:=x = -1 .. 1, y = -7*sqrt(1-x^2) .. 0;
Ax:=diff(A,x); Ay:=diff(A,y);Bx:=diff(B,x); By:=diff(B,y);
```

1. In this problem we wish to find and classify all the critical points of the expression (A).

(a) First obtain a rough idea of what this function looks like, by plotting it over the region $-1.5 \leq x \leq 1.5$, $-7 \leq y \leq 1$. You should be able to construct the graph without any hints, but be sure to use the `title` option to label it. Examine the graph for possible critical points, but keep your observations to yourself. The results that you calculate later may turn out to be different from what you expect.

(b) Now use Maple's `solve` command to find all the critical points of the expression (A) using the instruction `solA:=solve({Ax,Ay},{x,y});` Note that, if you ask *Maple* to solve an *expression* or a set of expressions, it assumes that you want to solve the equations in which all expressions are equal to zero. It is also a good idea is to name the solutions when they are computed, so we assign the result the name `solA`. The solutions obtained from the `solve` command are given to you as an *expression sequence*, i.e., several quantities separated by commas (as in the definition of $\mathcal{R}$ above). In this case, the elements in the sequence are sets of assignments of the variables. Then you can retrieve individual solutions using the usual *Maple* indexing convention, e.g., `solA[3]` refers to the third one. To check this, you can evaluate both `Ax` and `Ay` at `solA[3]` by typing `subs(solA[3],[Ax,Ay]);`. The brackets signify a *list* of quantities (the same structure that we used to represent vectors in Lab 1), and the substituted values will appear in the same order in the output as the expressions did in the input. The "seed file" contains this example, with the warning that it should be removed after you try it. Before you do, make sure you understand why the result is *not* surprising.

(c) Again using Maple's `diff` command, evaluate the second derivatives of (A). The names `Axx`, `Axy`, `Ayx`, and `Ayy` are reasonable choices, and are used in the prepared worksheet. Using this convention, `Axy` and `Ayx` are obtained by different computations, but they should be equal. The book (see p. 940) classifies critical points using a second derivative test involving a quantity

$$D = \frac{\partial^2 A}{\partial x^2} \cdot \frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 A}{\partial x\, \partial y} \cdot \frac{\partial^2 A}{\partial y\, \partial x}.$$

You are encouraged to *try to* call this expression D. It will lead to an error message, and you can find an explanation of that message using the help facility. When you find it, add a comment to your worksheet. Except for this comment, remove all evidence of the error from the final worksheet, and compute this expression D (using a different name) and evaluate it, together with anything else that you need to classify the critical point, at each critical point. Your report should also include **text** describing the information given by this second derivative test. When you have these values, add a line or two of text giving the type of each critical point.

2. Now consider the expression B. If you don't have the graph from Lab 0 handy, recreate it, but don't include it in your report. You can use a separate "scratch" worksheet for things that you don't want to include in your report. All definitions made during the session will be known on that sheet. (This works the other way also, so be careful. Something done on your scratch sheet may introduce a change that affects your main sheet. Your report is expected to be consistent, so it should be checked by selecting "Execute Worksheet" from the Edit menu in a fresh session.)

(a) Try to find critical points using the `solve` command as in problem 1. Insert a **text** description of your interpretation of the result.

(b) Now use the `fsolve` command to find the critical points which lie in the region $-2 \leq x \leq 2$, $-2 \leq y \leq 2$. There are exactly four of them, as indicated by the plot from Lab 0. Also find the values of the function the expression (B) at all of these critical points; be sure that your worksheet makes it clear which values are obtained at which points. The `fsolve` command uses an iterative method (such as Newton's method) to find the roots and sometimes the method does not converge to a root. You should consult the help file for the `fsolve` command to find how to restrict the search for a root to a smaller region, and use your plot to identify suitable regions. The seed file contains the line `p1:= fsolve({Bx,By},{x,y}, x =-1..0, y=0..1);` that finds one of the critical points and assigns it the name `p1`.

(c) This function is close to zero if $x^2 + y^2$ is large, and the plot reveals that it takes both positive and negative values. The maximum and minimum must be attained at critical points, and you should now know, and have names for, all of them to reasonable accuracy. Use this to determine the absolute minimum and absolute maximum values of the expression (B). Summarize in **text**.

3. In this problem, we find the absolute minimum and absolute maximum of the expression (A) of Problem 1 on the semi-ellipse $\mathcal{R}$ whose description was given at the beginning of this Lab (and included in the seed file). We know from the general theory that the absolute minimum and maximum of the expression A occur either (1) at critical points of the expression B which lie in the interior of the region $\mathcal{R}$ or (2) on the boundary of the region $\mathcal{R}$. The boundary consists of two smooth arcs to which the method of *Lagrange Multipliers* applies and two *corners* $(-1, 0)$ and $(1, 0)$ that must be considered separately.

(a) Obtain a plot of the expression A with the domain restricted to the region $\mathcal{R}$. The seed file contains an instruction that will produce the plot and supply a title. You should adjust the view to obtain a plot that will guide the determination of the extreme values and the points at which they are attained.

(b) Determine which of the critical points found in Problem 1 lie in the region $\mathcal{R}$ and evaluate the expression A at these points. State your conclusions in a **text** section, but leave the calculations supporting your conclusion in the worksheet. Be sure that you check both of the conditions $y \leq 0$ and $x^2 + y^2/49 \leq 1$ used to define $\mathcal{R}$.

(c) The restriction of A to the edge where $y = 0$ has critical points where $A_x(x, 0) = 0$, since this is just an interpretation of a single variable result. Obtain this expression and find where it is zero. Also consider the endpoints $(\pm 1, 0)$ at this time. Find the value of A at all of these points.

(d) For functions of two variables, the criterion for a extreme value of $f(x, y)$ subject to the constraint $g(x, y) = 0$ specializes to $f_x g_y = f_y g_x$. Solving this simultaneously with $g(x, y) = 0$ determines the points that must be considered. Use the `solve` command to find the solution to these two equations in x and y. The result will be an *algebraic* solution that is not too useful for our purpose. Indeed, (as of *MapleV*, release 5) there is no easy way to find all real solutions of numerically from this algebraic solution.

(e) The ellipse $x^2 + y^2/49 = 1$ can be parameterized by $x = \cos t$, $y = 7 \sin t$, and the arc we are interested in corresponds to $-\pi \leq t \leq 0$. The commands

```
Ael:=subs(x=cos(t),y=7*sin(t),A);
At:=diff(Ael,t);
plot(At,t=-Pi..0);
```

will restrict the function to the arc, differentiate with respect to the parameter t, and plot the derivative for the appropriate values of t so you can see where the roots of the derivative are. You can the use `fsolve` to find the individual roots of `At` and substitute these values into `Ael` to find the values of A at the points described by that value of the parameter. You now have all candidates for maxima and minima on this section of the boundary.

(f) Combine the results of (b), (c), and (e). You have a list of all possible locations of maxima and minima on $\mathcal{R}$ and the value of A at these points. By identifying the smallest and largest values of A in this list, you will have found the extreme values on $\mathcal{R}$ and point where those values are attained. Use **text** to state your conclusions.

End of Lab 3