Math 251:07–09 — Spring 2001
TF3   PH-111
Prof. Bumby
Lab 4   SURFACES

This description of the lab is only for the sections identified in the header. Other sections will use a different version. A *seed file* `lab4.mws` for this version can be downloaded from the web page for this section. That file will assure that the descriptions given here are accurately transcribed to your worksheet.

Please turn in only the printout of your Maple worksheet(s). Use the **text** feature of Maple to add a header containing your name, identify the separate parts of the lab, and give **explicit answers to all questions asked** (do not write this material in by hand). Remove from the worksheet any extraneous material and any errors you have made. To be sure that the input and output shown agree, you should use a `restart` command and then select **Execute Worksheet** from the Edit Menu.

Section 0. Introduction. In addition to the `plots` library that you load using the `with(plots)` instruction, the basic operations with vectors will be needed. The seed file repeats the definitions from Lab 1. A new function is also introduced here that will be used to compute surface integrals.

```
scalP:=(sc,vec)->map(x->sc*x,vec);
dotP:=(v1,v2)->sum(v1[i]*v2[i],i=1..nops(v1));
crossP:=(v1,v2)->[v1[2]*v2[3]-v1[3]*v2[2],
        v1[3]*v2[1]-v1[1]*v2[3],v1[1]*v2[2]-v1[2]*v2[1]];
proJ:=(a,b)->scalP(dotP(a,b)/dotP(a,a),a);
surfint:=(field,surface,NdS,lim1,lim2)->
        int(int(dotP(field(surface),NdS),lim2),lim1);
V1:=pt->[0,0,pt[3]];V2:=pt->[0,pt[2],0];
```

These definitions illustrate the ability to extend *Maple* by defining new functions. The functions defined here are all *one-liners*, in the sense that they are given by a single formula, so that the definition serves mainly to assure that the same computation will be used without requiring the user to enter (or copy) a complicated statement. More elaborate computations, such as those fount in the `plots` library, can be built as **procedures**. In all cases, the arguments of function are arranged as an **expression sequence**, i.e., a collection of expressions separated by commas. It is possible to have a single name represent such an expression sequence, and its use among the arguments of a function will supply several consecutive arguments. For example, in the `surfint` function, the two arguments giving the limits of integration may be given by a single name giving the domain. Indeed the reason for integrating first with respect to `lim2` is to allow the same domain to be used in a `plot3d` command. For this course, it is convenient to represent a vector as a **list**, which is a single object that is a list enclosed in square brackets. Both lists and names representing expression sequences allow individual terms to be extracted by following the object by an index in square brackets (as in the use to `v1[3]` to describe the third entry of the list `v1`.

In some places, it is useful to express the direction of a vector by a unit vector. Since *Maple* does its computations with complex numbers wherever possible, it often fails to simplify square roots of expressions that are obviously squares. You can find the square of the length of `v` as `dotP(v,v)`, but you may need to manually define a variable `lenv` representing its length. However you obtain `lenv`, a unit vector in the direction of `v` is then given by `scalP(1/lenv,v)`. The result of this computation will be a *list*. You get the expected result if add lists, and the `simplify` command automatically simplifies all entries in a list. However, multiplication by a scalar expression is not usually done elementwise without the `map` operation, or the `scalP` function defined in terms of it. Thus, if the result of a calculation is difficult to interpret, it is probably wrong, because some computation failed to give a simple list.

Another structure that is used in plots and integrals is a **range**, which consists of two expressions separated by a pair of dots.

The arguments of `surfint` are a **function** taking one **list** to another **list** representing the vector field to be integrated (the examples `V1` and tt V2 are included), a **list** giving a parametric description of the surface, another **list** giving a normal vector whose length is the density of surface area in the parameter space, and **ranges** giving the limits of integration arranged so that the second one may depend on the parameter named in the first range, but first one is numerical.

Section 1. Graphs of functions. Consider the paraboloid

$$z = 8 - y^2 - 2x^2 \tag{1}$$

and the parabolic cylinder

$$z = y^2 \tag{2}$$

Since these expressions are used in several ways, it will be useful to begin by introducing the names `z1` and `z2` for the expressions on the right sides of (1) and (2), respectively. Such expressions are included in the seed file. The point over $(x, y) = (0, 0)$ in (1) has $z = 8$ and is a global maximum, while in (2), $z = 0$ which is a minimum (along with the all the other points on the x-axis). Thus, the region between these surfaces is bounded above by (1) and below by (2), so that the **outward** orientation on the boundary will be **upward** on (1) and **downward** on (2).

(a) For a first view of this figure, plot the two surfaces given above on the same set of axes for $-2 \le x \le 2$, $-\sqrt{8} \le y \le \sqrt{8}$. The basic command is given in the seed file, but you should add a title.

(b) To set up integrals, it is necessary to know the values of (x, y) that give points of the region, that is those with $z1 \le z2$. This will be the interior of the region with $z1 = z2$. Again, you are given the `implicitplot` command, but you must add a title. You should also try the plot without the `scaling=CONSTRAINED` option.

Then, use your knowledge of the standard equations of *conic sections* (Section 10.6 of the textbook) to identify the curve. Insert **text** identifying the curve, including the coordinates of its *vertices*.

(c) Now, use an instruction of the form `region1:=y=a..b,x=c..d;` with appropriate values of `a, b, c, d`. In particular, `a` and `b` should be constants, while `c` and `d` should be functions of y. Test your definition using the instructions in the seed file that produce a graph of $z1$ and $z2$ over this region. The result should look like the boundary of a solid body. However, contrasting colors have been used to identify the upper and lower boundaries of the region. Do not continue until you are convinced by this graph that your definition of `region1` is correct.

(d) We are now ready to find the volume of figure. The instructions

```
top:=[x,y,z1],[-diff(z1,x),-diff(z1,y),1];
bottom:=[x,y,z2],[diff(z2,x),diff(z2,y),-1];
surfint(V1,top,region1)+surfint(V1,bottom,region1);
```

constructs **expression sequences** `top` and `bottom` giving points and outward normals in terms of x and y for these functions. The sum of the surface integrals of the vector field `V1` is a fancy way of expressing the integral of the difference of the values of z on these surfaces with respect to area in the xy plane. It appears in exactly this form in the proof of the divergence theorem, corresponding to the first step in the evaluation of volume as a triple integral.

The vector field `V2` arises in a similar way when the volume is found by integrating first with respect to y. The advantage of using surface integrals is that it is not necessary to have an explicit expression of y as a function of x and z on different parts of the surface. Indeed, the same description of the surface using x and y as independent variables can be used to integrate any vector field because we have used the notion of **outward orientation** to replace the use of **top** and **bottom** that is required when the quantities were first formulated as double integrals for figures bounded by graphs of functions.

Verify that the integration of `V2` over the surface gives the same answer as obtained by integrating `V1`.

(e) Now find two different vector fields whose integral computes the first moment with respect to the xy-plane. In addition to evaluating the integrals and verifying that the answers are the same, include a **text** statement describing how these surface integrals arise from integrals with respect to two coordinates.

Section 2. A torus. The parametric description of surfaces can be built up by following a description of the construction of the surface. We next investigate a torus with the following description. From the point $(a, 0, 0)$ take a line L parallel to the z-axis. Our figure will be obtained by rotating a circle about L. More precisely, we want the intersection of our surface with a plane containing L to be a circle of radius c whose center is at distance b from L and in the xy-plane. The values of a, b and c will be fixed later, To parameterize the surface, one parameter should select a plane containing L and a second should describe a circle in this plane. The line L contains the point $a(a, 0, 0)$ and has direction $\mathbf{k} = \langle 0, 0, 1\rangle$. A vector perpendicular to L has direction $\mathbf{v}_u = \langle \cos(u), \sin(u), 0\rangle$ for some u. This latter vector gives the direction from $(a, 0, 0)$ to the center of the rotating circle. From this center, we reach a point on the rotating circle by going a distance c along an arbitrary vector. The directions in this plane are expressed in terms of $\mathbf{k}$ and $\mathbf{v}_u$, and the unit vectors can be taken to be $\cos(v)\mathbf{v}_u + \sin(v)\mathbf{k}$ for some v. The complete surface will have u and v each taking values between 0 and 2π. The specific torus we want to consider has $a = 3/2$, $b = 1/2$ and $c = 1/4$. Instructions to describe the surface are

```
a:=3/2;b:=1/2;c:=1/4;
P0:=[a,0,0];
diru:=[cos(u),sin(u),0];
kk:=[0,0,1];
torus:=P0+scalP(b,diru)+scalP(c*cos(v),diru)+scalP(c*sin(v),kk);
```

To define surface integrals on this surface, find the partial derivatives of `torus` with respect to u and v to get the tangents to the *grid curves* and compute the cross product of these two vectors. You should be able to get *Maple* to this without further hints. As a check, the length of this vector should be found to be $(\cos(v) + 2)/16$. Perform calculations to check this and describe the results in **text**. When you have written this description, you can remove any calculation of the length of the vector from the worksheet, although the vector itself should be saved.

Since this vector is never zero, it gives a consistent orientation over the whole surface, and checking at the point where $u = v = 0$ should reveal the it is the **outward** orientation.

Section 3. Biting the donut. Our goal is now to describe the intersection of the interior of the torus of Section 2 with the interior of the sphere $x^2 + y^2 + z^2 = 1$.

(a) Begin by plotting both surfaces. In order to see inside the sphere, we plot only the portion with $x > 0$. Execute the following commands.

```
A:=plot3d(torus,u=0..2*Pi,v=0..2*Pi):
B:=implicitplot3d(x^2+y^2+z^2=1,x=0..1,y=-1..1,z=-1..1):
display({A,B},title="bite the donut");
```

Our goal is to remove the portions of these surfaces that are not part of the exact boundary of the intersection of the interiors. Such a description can be used to find integrals over the boundary that will express geometrical or physical properties of the intersection, as was done for the surfaces in Section 1.

(b) The curve of intersection can be found by substituting the parametric description of the torus into the implicit equation of the sphere. Do this, and simplify the result to get an equation in the parameters. You should get an equation that can be solved for $\cos(u)$ in terms of $\cos(v)$. You may need to experiment to find the conditions that are consistent with the requirement the $-1 \leq \cos(u) \leq 1$ and $-1 \leq \cos(v) \leq 1$. When this is done, you will be able to find a symmetric interval around 0 in v for which there are points on the curve of intersection. For each such v values of $\cos(u)$ can be found. The values of u having this cosine are in the second and third quadrants, and the usual arccosine function gives the values in the second quadrant. For each v the values of u will be an interval symmetric about π whose left endpoint has just been found. Although the function is not simple, this leads to an explicit description of the portion of the torus inside the sphere.

To get the portion of the sphere bounded by this curve, the values of $\sin(v)$ on the torus are converted to values of z, and then the values of $\sin(u)$ are converted to values of y. For these values, the portion of the sphere that we need has $x = \sqrt{1 - y^2 - z^2}$, so is the graph of a function of y and z.

You may not be able to achieve a complete determination of the boundary. Use **text** to summarize any partial results.