In elementary geometry, you learn that a triangle is determined by the lengths of its sides. In trigonometry, the law of cosines is derived to show that the angle θ between sides a and b and opposite side c is determined by

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If we use vectors $\mathbf{a}$ and $\mathbf{b}$ for the vectors pointing away from the vertex C, then a vector along the third side is $\mathbf{a} - \mathbf{b}$, and

$$c^2 = |\mathbf{a} - \mathbf{b}|^2 .$$

If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then

$$\begin{aligned} c^2 &= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\ &= a_1^2 - 2a_1b_1 + b_1^2 \\ &+ a_2^2 - 2a_2b_2 + b_2^2 \\ &+ a_3^2 - 2a_3b_3 + b_3^2 \\ &= a^2 + \underline{\qquad\qquad} + b^2 \end{aligned}$$

Comparing this with the law of cosines, gives

$$ab\cos\theta = a_1b_1 + a_2b_2 + a_3b_3.$$

The expression on the right is taken as the definition of the **dot product** $\mathbf{a}\cdot\mathbf{b}$, and the geometric interpretation is a theorem. The form of the definition shows that the dot product is *linear* in each factor. A special case of the definition shows that $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$, and the Pythagorean theorem shows that nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a}\cdot\mathbf{b} = 0$. It is convenient to extend the use of the word "perpendicular" so the a zero vector is considered perpendicular to all vectors.

Projections and components. We have the formulas

$$\begin{aligned} \operatorname{comp}_{\mathbf{a}}\mathbf{b} &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} \\ \operatorname{proj}_{\mathbf{a}}\mathbf{b} &= \left(\operatorname{comp}_{\mathbf{a}}\mathbf{b}\right)\frac{\mathbf{a}}{|\mathbf{a}|} \\ &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} \end{aligned}$$

The cross product. The formula summarized by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{$*$}$$

clearly gives a vector perpendicular to

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \text{ and } \mathbf{b} = \langle b_1, b_2, b_3 \rangle,$$

using a systematic formula. It is zero when $\mathbf{a} \parallel \mathbf{b}$, and only in this case. Direct computation (Theorem 6) shows that the length of this vector is the product of the lengths of **a** and **b** and the sine of the angle between them. This shows that the length is the area of the parallelogram spanned by **a** and **b**. The only surprise is that interchanging the roles of **a** and **b** sends the product into its negative. Formula ($*$) defines the cross product $\mathbf{a} \times \mathbf{b}$. In addition to giving directions perpendicular to two given directions and finding areas, it gives a computational way to describe **orientation** in space — the distinction between *left* and *right*, because it is **skew commutative**.

This product is also **linear** in each factor. There are also rules for interpreting triple product — items 5 and 6 of theorem 8 that we won't do much with.

Exercises 12.3

Compute $\mathbf{a} \cdot \mathbf{b}$.

$$\mathbf{a} = \langle 4, -1 \rangle \quad \mathbf{b} = \langle 3, 6 \rangle \tag{3}$$

$$\mathbf{a} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \quad \mathbf{b} = 5\mathbf{i} + 9\mathbf{k} \tag{7}$$

$$\mathbf{a} = \mathbf{i} + \mathbf{k} \quad \mathbf{b} = \mathbf{j} + \mathbf{k} \tag{Z}$$

Method: Use definition.

Find angle between **a** and **b** for above examples and

$$\mathbf{a} = \langle 3, 4 \rangle \quad \mathbf{b} = \langle 5, 12 \rangle \tag{15}$$

$$\mathbf{a} = \mathbf{j} + \mathbf{k} \quad \mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k} \tag{19}$$

Method: Compute by definition; then use geometric interpretation.

Find the scalar and vector projections of **b** on **a**.

$$\mathbf{a} = \langle 2, 3 \rangle \quad \mathbf{b} = \langle 4, 1 \rangle \tag{39}$$

$$\mathbf{a} = \langle 4, 2, 0 \rangle \quad \mathbf{b} = \langle 1, 1, 1 \rangle \tag{41}$$

Method: Use definitions.

Exercise 63 Interpret and prove

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2\,.$$

Method: For interpretation, draw a picture starting from general **a** and **b** with other vectors constructed. For proof, carefully use $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$ and linearity of dot product.

Exercises 12.4

Compute $\mathbf{a} \times \mathbf{b}$.

$$\mathbf{a} = \langle 1, 2, 0\rangle \quad \mathbf{b} = \langle 0, 3, 1\rangle \tag{1}$$

$$\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} \quad \mathbf{b} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k} \tag{7}$$

Method: Use determinant description; or expand by linearity and employ the multiplication table for **i**, **j** and **k**.

#25. Find vector orthogonal to plane through

$$P(1, 0, 0,), \quad Q(0, 2, 0), \quad R(0, 0, 3)$$

and the area of $\triangle PQR$.

Method Introduce vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, and relate desired quantities to properties of cross product.