Equation of line. A line in space is determined by a point $P_0(x_0, y_0, z_0)$ and a direction $\mathbf{v} = \langle a, b, c\rangle$. Once you have the six numbers, all that is needed to get the equation is to write them in the right places. Things are slightly complicated by three ways of writing the equation being in common use. The keywords identifying these forms must be known.

Vector form. The form closest to the geometry of the line uses the idea that vectors in the same direction are related by being scalar multiples of one another. Thus, the vector $\overrightarrow{P_0P}$ from the given point P_0 to the variable point $P(x, y, z)$ should be a multiple of the vector $\mathbf{v}$. The scalar multiple is usually denoted by t, although different letters should be used for different lines in the same problem. Changing the parentheses around the quantities in P_0 and P to angle brackets transforms them into the vectors $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$, and the vector equation becomes

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Parametric form. The vector equation is an abbreviation for the three scalar equations obtained by equating the components. For the data given above, this is

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

Symmetric form. Each of these equations can be solved for the parameter t. The point P lies on the line if you always get the same answer. This puts the equation in a form that emphasizes a test for whether P lies on the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

This may even be used if some of the coefficients a, b or c are zero. The other forms show that this should be interpreted as requiring that a zero denominator requires the corresponding numerator to be zero.

How else could a line be given? Another way to give a line is by specifying two points P_0 and $P_1(x_1, y_1, z_1)$ on the line. This is immediately converted into the original form by taking $\mathbf{v} = \overrightarrow{P_0P_1}$.

A line may also be given as an intersection of two planes. We will return to this after discussing how planes are given.

Parallel lines; intersections of lines; skew lines. Lines are said to be parallel if they have the same direction. Thus, $P_0 + t\mathbf{v}_0$ is parallel to $P_1 + u\mathbf{v}_1$ if there is a constant λ such that $\mathbf{v}_0 = \lambda\mathbf{v}_1$. The *inclusive* form of this definition means that *identical* lines are considered to be parallel. In our example, the lines are identical if $\overrightarrow{P_0P_1}$, $\mathbf{v}_0$, and $\mathbf{v}_1$ all have the same direction. Although the same line may have two equations that appear different, it is easy to check whether different equations represent the same line.

Two lines intersect if you can find t and u so that

$$P_0 + t\mathbf{v}_0 = P_1 + u\mathbf{v}_1.$$

In scalar form, this gives three equations relating the

two variables t and u. This suggests that pairs of lines do not usually intersect. A simple example is

$$t\langle 1, 0, 0\rangle \text{ and } (0, 0, 1) + u\langle 0, 1, 0\rangle.$$

Lines that are not parallel and do not intersect are called *skew*.

The equation of a plane. A single equation

$$ax + by + cz = d \qquad (*)$$

defines a plane. To see this, suppose you have one solution (x_0, y_0, z_0) of (*), which could be obtained by setting $x = x_0$ and $y = y_0$ arbitrarily and solving $(*)$ for z to find z_0. Then $(*)$ is equivalent to

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \qquad (**)$$

This says that the general vector in the plane $(**)$ $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n} = \langle a, b, c\rangle$. This leads to a geometric way of recognizing the solutions of $(*)$ as a plane. It also says that the equation can be written if one knows a point in the plane and a direction perpendicular to the plane.

The plane through three points. A common way to describe a plane is to give three points P_0, P_1, P_2 in the plane. Then P_0P_1 and P_0P_2 are two directions in the plane and $P_0P_1 \times P_0P_2$ is perpendicular to the plane. This, together with the point P_0 in the plane gives the information needed to write the equation.

Intersection of line and plane. In general, an equation gives a condition on the coordinates (x, y, z) of a general point for it to lie in a set. The intersection of a line, given in parametric form, and a plane, given by an equation, is found by substituting the parametric description of the points on the line into the equation for the plane. This gives a single equation in the parameter. Solve this equation and use that value in the description of the line to find the coordinates of the intersection.

Intersection of two planes. The easiest way to find the line of intersection of two planes is to use the geometric interpretation of the equation of a plane. The coefficients give a direction perpendicular to the plane, i.e. perpendicular to each direction in the plane. The direction of the line of intersection is thus perpendicular to the two vectors giving the directions of the planes, so its direction is given by the cross product. It remains to find a point on the line. This can be done by choosing the z-coordinate arbitrarily ($z = 0$ is a good choice). The equations of the planes then give two equations in x and y that are easy to solve simultaneously.

Exercises

#3. Find line through $(-2, 4, 10)$ and parallel to $\langle 3, 1, -8\rangle$.

Method. Reformat given numbers.

#7. Find line though points $(3, 1, -1)$ and $(3, 2, -6)$.

Method. First, find direction.

#11. Show that the line through $(2, -1, -5)$ and $(8, 8, 7)$ is parallel to the line through $(4, 2, -6)$ and $(8, 8, 2)$.

Method. Determine directions.

#15. Determine whether

$$\frac{x-4}{2} = \frac{y+5}{4} = \frac{z-1}{-3} \qquad (L_1)$$

$$\frac{x-2}{1} = \frac{y+1}{3} = \frac{z}{2} \qquad (L_2)$$

are parallel, skew or intersecting.

Method. Determine directions.

#23. Find the equation of the plane through the origin and parallel to $2x - y + 3z = 1$.

Method. Reformat given numbers.

#27. Find the equation of the plane through $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$.

Method. Find directions in the plane, then perpendicular one to the plane.

#31. Find the equation of the plane through $(6, 0, -2)$ that contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$.

Method. Begin by finding directions in the plane.

#35. Find the point where the the line $x = 1 + t$, $y = 2t$, $z = 3t$ meets the plane $x + y + z = 1$.

Method. Find the value of t that locates the point on the line.