

**Vector functions.** We consider functions from  $\mathbb{R}$  to  $\mathbb{R}^3$ . The appropriate notion of continuity of such functions is that each component be continuous. Similarly, to differentiate such a function, use the derivatives of the components as the components of a vector. These properties are not arbitrary: they can be proved from a general definition of limit. If the values of a vector function are plotted, one gets a **space curve**. The simplest example is a linear function, whose graph is a line. The **parametric equations** of plane curves considered in Chapter 10 are also special cases of this definition. Several other examples appear in the text. An interesting example is the **helix**, given by  $\langle \cos t, \sin t, t \rangle$ . On the surface where  $x^2 + y^2 = 1$ , this curve moves evenly around the cylinder and up the axis. Examples appearing in the exercises for Section 13.1 are

$$x = t \quad y = \frac{1}{1+t^2} \quad z = t^2 \quad (9)$$

$$x = \cos t \quad y = \sin t \quad z = \sin 5t \quad (11)$$

$$x = \sin t \quad y = 3 \quad z = \cos t \quad (17)$$

**Derivatives.** If  $\mathbf{r}(t)$  is a vector function, then

$$\mathbf{r}(t+h) - \mathbf{r}(t)$$

is a difference of vectors for each  $t$  and  $h$ , so it is the vector whose components are  $x(t+h) - x(t)$ ,  $y(t+h) - y(t)$  and  $z(t+h) - z(t)$ . Dividing the vector difference by the scalar  $h$  involves dividing each component by  $h$ , but it also admits a geometric description as a rescaling of the vector difference. For reasonable functions  $x(t)$ ,  $y(t)$ , and  $z(t)$ , the difference quotients have a limit which is the derivative. This means that rescaled difference of vectors also has a limit which is the vector of derivatives. We describe the direction of the vector here, and its length will be interpreted in the next section. The difference  $\mathbf{r}(t+h) - \mathbf{r}(t)$  is a chord of the curve, and the rescaling simply replaces the chord by a vector in the same direction whose length is (usually) bounded away from zero as  $h \rightarrow 0$  for fixed  $t$ . The limit of these directions is the direction of the limit. As with plane curves, such a limit of chords is what we expect to be a **tangent direction** to the curve at the point  $\mathbf{r}(t)$ . In particular, one can

scale the direction to obtain the **unit tangent vector**  $\mathbf{T}$ .

A **tangent line** to the curve at this point is the line through  $\mathbf{r}(t)$  in the direction of  $\mathbf{r}'(t)$ . Usually, a numerical value of  $t$  will be given (perhaps indirectly by specifying the value of  $\mathbf{r}(t)$ ) which will be substituted into these expressions (after finding  $\mathbf{r}'(t)$  from the function  $\mathbf{r}(t)$ ).

The calculus of this derivative has all of the expected properties: the derivative of a sum is the sum of the derivative and the derivative of a scalar constant multiple is that multiple of the derivative. Also, all of the products that we have met are built in some way from products of a component of the first factor times a component of the second factor. This can be used to prove that all analogs of the product rule hold. There is also a chain rule for  $\mathbf{r}(t(u))$ .

## Exercises 13.2

All exercises are some variation of finding the derivative, sometimes scaled to give  $\mathbf{T}$ . In some problems, the numerical description of a point on the curve is given, allowing numerical values of these quantities and a tangent line and a tangent line to be found.

$$\mathbf{r}(t) = \langle t^2, 1 - t, \sqrt{t} \rangle \quad (9)$$

$$\mathbf{r}(t) = \langle 1, -1, e^{4t} \rangle \quad (11)$$

$$\mathbf{r}(t) = \langle t, 2 \sin t, 3 \cos t \rangle (t = \pi/6) \quad (19)$$