Arc length. If the vector function $\mathbf{r}(t)$ is thought of as giving position as a function of time, then its derivative $\mathbf{r}'(t)$ gives **velocity**. The length of $\mathbf{r}'(t)$ measures the **speed** and we shall see that the distance traveled along the curve is the integral of speed. We have already met $\mathbf{T}$, which is a unit vector in the direction of $\mathbf{r}'(t)$. If a different parameter u is used to describe the curve, with t being an increasing function of u, $d\mathbf{r}/du = (dt/du)(d\mathbf{r}/dt)$. The first factor is a scalar, so it does not affect T.

The usual approach to measuring the length of a curve $\mathbf{r}(t)$ between the point where $t = a$ and the point where $t = b$ is to select values $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ and find the length of the polygonal path connecting the points $\mathbf{r}(t_i)$ in order. This gives a sum of terms of the form

$$\Delta t \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}.$$

With any kind of luck, this will approach

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

as the t_i get closer together. We do not prove this formula, but we accept it as sufficiently plausible to be taken as a definition of arc length.

One encouraging fact is that, when the parameterization is changed by the substitution $t = g(u)$ with a monotonic function g, the integral doesn't change. That is, our way of finding the length of the curve depends on the curve and not on how it is drawn.

Exercises. There are only a few special curves for which this integral can be evaluated in closed form. Note how the examples here simplify.

$$x = 2\sin t \quad y = 5t \quad z = s\cos t \tag{1}$$

$$x = \sqrt{2}t \quad y = e^t \quad z = e^{-t} \tag{3}$$

Parameterization by arc length. As long as we know that a function is defined, whether or not we have previously named it, is is available for use. The arc length integral always gives arc length s as a function of the original parameter t. The derivative ds/dt is the integrand of the arc length integral, which is always positive. Thus, s is an increasing function of t, and there is an inverse function giving t in terms of s. This parameterization is often used to give a geometric definition of a quantity that we intend to study. In order to compute such quantities, a substitution is made to express it in terms of the parameter t appearing in the original definition of the curve. In particular, $\mathbf{T}$ is the derivative of the position vector with respect to arc length.

A useful result. The usual rules of calculus for sums and products are easily proved for derivatives of vectors. One consequence of this is that if $\mathbf{a}(t)$ is of constant length, so that $\mathbf{a}(t) \cdot \mathbf{a}(t)$ is a constant function, then

$$0 = \mathbf{a}(t) \cdot \mathbf{a}'(t) + \mathbf{a}'(t) \cdot \mathbf{a}(t) = 2\mathbf{a}(t) \cdot \mathbf{a}'(t)$$

so that $\mathbf{a}'(t)$ is always perpendicular to $\mathbf{a}(t)$.

Normals and curvature. From the last result, we get that $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$. The direction of $\mathbf{T}'(t)$ is called the **principal normal** and denoted $\mathbf{N}$. Changing the parameter multiplies the derivative of $\mathbf{T}$ by a scalar (positive if the parameters are increasing functions of one another), so $\mathbf{N}$ is independent of the parameterization. If you take *arc length* as the parameter, then the magnitude of the derivative is also significant. This value is called **curvature**, and denoted κ, here described by **definition (8)**. Finally, it is not actually necessary to construct this parameterization, since the value at any point can be found from the chain rule. This gives **formula (9)**, which we use in the example below. However, this gives all geometric features as functions of the original parameter.

The main example. If $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, describing a circle of radius a in the xy plane, $\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$, and we can *see* its length and direction: $ds/dt = a$ and $\mathbf{T} = \langle -\sin t, \cos t\rangle$. Then

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{\langle -\cos t, -\sin t\rangle}{a}$$

in this case. Geometrically, we see that $\mathbf{N}$ is a unit vector pointing towards the center of the circle, and $\kappa = 1/a$.

Another curvature formula. Another formula for curvature differentiates $\mathbf{r}'(t)$ directly without rescaling it into $\mathbf{T}(t)$. This appears as **Theorem (10)**.

$$\kappa(t) = \frac{\left|\mathbf{r}'(t) \times \mathbf{r}''(t)\right|}{|\mathbf{r}'(t)|^3}.$$

The proof is direct. The product rule gives

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'.$$

Since $\mathbf{T} \parallel \mathbf{r}'(t)$, the first term contributes nothing when you take the cross product with $\mathbf{r}'(t)$, and

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^2 \mathbf{T} \times \mathbf{T}'.$$

Finally, $\mathbf{T} \perp \mathbf{T}'$, so the length of the cross product is the product of the lengths, and $\mathbf{T}$ is a unit vector,

while the length of $\mathbf{T}'$ (from (9)) is $\kappa(ds/dt)$. This formula is most useful if ds/dt is complicated.

The curvature of plane curves is often treated as a signed quantity, positive if the curve bends to the left and negative if it bends to the right. However, orientation in space requires three vectors. If you just have a first and second vector, the “right hand rule” tells how to find a third vector perpendicular to both of then such the the ordered triple has positive orientation. This leads to the custom of requiring κ to be positive and defining the direction of the principal normal by

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}.$$

The vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, called the **binormal**, completes an oriented orthonormal basis.

Exercises. Some exercises for finding $\mathbf{T}$, $\mathbf{N}$ and κ are

$$x = \tfrac{1}{3}t^3 \quad y = t^2 \quad z = 2t \tag{13}$$

$$x = \sin t \quad y = \cos t \quad z = \sin t \tag{17}$$

We will examine the use of both formulas 9 and 10 for finding κ.