

Arc length. If the vector function $\mathbf{r}(t)$ is thought of as giving position as a function of time, then its derivative $\mathbf{r}'(t)$ gives **velocity**. The length of $\mathbf{r}'(t)$ measures the **speed** and we shall see that the distance traveled along the curve is the integral of speed. We have already met \mathbf{T} , which is a unit vector in the direction of $\mathbf{r}'(t)$. If a different parameter u is used to describe the curve, with t being an increasing function of u , $d\mathbf{r}/du = (dt/du)(d\mathbf{r}/dt)$. The first factor is a scalar, so it does not affect T .

The usual approach to measuring the length of a curve $\mathbf{r}(t)$ between the point where $t = a$ and the point where $t = b$ is to select values $a = t_0 < t_1 < t_2 < \dots < t_n = b$ and find the length of the polygonal path connecting the points $\mathbf{r}(t_i)$ in order. This gives a sum of terms of the form

$$\Delta t \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2 + \left(\frac{\Delta z}{\Delta t}\right)^2}.$$

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Parameterization by arc length. As long as we know that a function is defined, whether or not we have previously named it, it is available for use. The arc length integral always gives arc length s as a function of the original parameter t . The derivative ds/dt is the integrand of the arc length integral, which is always positive. Thus, s is an increasing function of t , and there is an inverse function giving t in terms of s . This parameterization is often used to give a geometric definition of a quantity that we intend to study. In order to compute such quantities, a substitution is made to express it in terms of the parameter t appearing in the original definition of the curve. In particular, \mathbf{T} is the derivative of the position vector with respect to arc length.

A useful result. The usual rules of calculus for sums and products are easily proved for derivatives of vectors. One consequence of this is that if $\mathbf{a}(t)$ is of constant length, so that $\mathbf{a}(t) \cdot \mathbf{a}(t)$ is a constant function, then

$$0 = \mathbf{a}(t) \cdot \mathbf{a}'(t) + \mathbf{a}'(t) \cdot \mathbf{a}(t) = 2\mathbf{a}(t) \cdot \mathbf{a}'(t)$$

so that $\mathbf{a}'(t)$ is always perpendicular to $\mathbf{a}(t)$.

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With any kind of luck, this will approach

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

as the t_i get closer together. We do not prove this formula, but we accept it as sufficiently plausible to be taken as a definition of arc length.

One encouraging fact is that, when the parameterization is changed by the substitution $t = g(u)$ with a monotonic function g , the integral doesn't change. That is, our way of finding the length of the curve depends on the curve and not on how it is drawn.

Exercises. There are only a few special curves for which this integral can be evaluated in closed form. Note how the examples here simplify.

$$x = 2 \sin t \quad y = 5t \quad z = s \cos t \quad (1)$$

$$x = \sqrt{2}t \quad y = e^t \quad z = e^{-t} \quad (3)$$

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Normals and curvature. From the last result, we get that $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$. The direction of $\mathbf{T}'(t)$ is called the **principal normal** and denoted \mathbf{N} . Changing the parameter multiplies the derivative of \mathbf{T} by a scalar (positive if the parameters are increasing functions of one another), so \mathbf{N} is independent of the parameterization. If you take *arc length* as the parameter, then the magnitude of the derivative is also significant. This value is called **curvature**, and denoted κ , here described by **definition (8)**. Finally, it is not actually necessary to construct this parameterization, since the value at any point can be found from the chain rule. This gives **formula (9)**, which we use in the example below. However, this gives all geometric features as functions of the original parameter.

The main example. If $\mathbf{r}(t) = \langle a \cos t, a \sin t \rangle$, describing a circle of radius a in the xy plane, $\mathbf{r}'(t) = \langle -a \sin t, a \cos t \rangle$, and we can see its length and direction: $ds/dt = a$ and $\mathbf{T} = \langle -\sin t, \cos t \rangle$. Then

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{\langle -\cos t, -\sin t \rangle}{a}$$

in this case. Geometrically, we see that \mathbf{N} is a unit

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vector pointing towards the center of the circle, and $\kappa = 1/a$.

Another curvature formula. Another formula for curvature differentiates $\mathbf{r}'(t)$ directly without rescaling it into $\mathbf{T}(t)$. This appears as **Theorem (10)**.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

The proof is direct. The product rule gives

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2} \mathbf{T} + \frac{ds}{dt} \mathbf{T}'.$$

Since $\mathbf{T} \parallel \mathbf{r}'(t)$, the first term contributes nothing when you take the cross product with $\mathbf{r}'(t)$, and

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^2 \mathbf{T} \times \mathbf{T}'.$$

Finally, $\mathbf{T} \perp \mathbf{T}'$, so the length of the cross product is the product of the lengths, and \mathbf{T} is a unit vector,

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while the length of \mathbf{T}' (from (9)) is $\kappa(ds/dt)$. This formula is most useful if ds/dt is complicated.

The curvature of plane curves is often treated as a signed quantity, positive if the curve bends to the left and negative if it bends to the right. However, orientation in space requires three vectors. If you just have a first and second vector, the “right hand rule” tells how to find a third vector perpendicular to both of them such that the ordered triple has positive orientation. This leads to the custom of requiring κ to be positive and defining the direction of the principal normal by

$$\frac{d\mathbf{T}}{ds} = \kappa \mathbf{N}.$$

The vector $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, called the **binormal**, completes an oriented orthonormal basis.

Exercises. Some exercises for finding \mathbf{T} , \mathbf{N} and κ are

$$x = 1/3t^3 \quad y = t^2 \quad z = 2t \quad (13)$$

$$x = \sin t \quad y = \cos t \quad z = \sin t \quad (17)$$

We will examine the use of both formulas 9 and 10 for finding κ .

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