Motion in Space. These formulas are used in mechanics. If $\mathbf{r}(t)$ represents the **position** of a body as a function of time, then $\mathbf{r}'(t)$ is **velocity** and $\mathbf{r}''(t)$ is **acceleration**. The key idea in Newton's explanation of motion was that motion represented the effect of **forces** and **force** is **mass** times **acceleration**. All of these quantities except **mass** are vectors; **mass** is a scalar that is constant for ordinary objects. If you observe the position function $\mathbf{r}(t)$, you determine the acceleration $\mathbf{r}''(t)$ and use that to help identify the force.

Exercises. No claim of physical significance for these exercises will be made. They serve only to illustrate the Calculus using the language of mechanics. The general instructions are to use given quantities to find all of **position** $\mathbf{r}(t)$, **velocity** $\mathbf{v}(t) = \mathbf{r}'(t)$, **speed** $v = ds/dt$, and **acceleration** $\mathbf{a}(t) = \mathbf{r}''(t)$ using the given information.

$$\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle \tag{9}$$

$$\mathbf{r}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle \tag{11}$$

$$\mathbf{a}(t) = \mathbf{k} \quad \mathbf{v}(0) = \mathbf{i} - \mathbf{j} \quad \mathbf{r}(0) = 0 \tag{15}$$

A more elaborate problem is #19, which gives the motion

$$\mathbf{r}(t) = \left\langle t^2, 5t, t^2 - 16t \right\rangle$$

and asks for the time at which the speed is largest.

Components of acceleration. The formula

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'.$$

was derived in Section 13.3 (page 852). We also have $\mathbf{T}' = \kappa(ds/dt)\mathbf{N}$, so

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$

Writing v in place of ds/dt gives formula (7) of Section 13.4. The quantity v represent the **speed** of the object. The vectors $\mathbf{T}$ and $\mathbf{N}$ are perpendicular unit vectors that are part of a coordinate system that moves with the object. In particular $\mathbf{T}$ is "straight ahead". In this coordinate system, the first part of the expression for $\mathbf{r}''(t)$ describes the part of the acceleration (and, hence, of the force) that leads to a change of speed, while the second part describes the part of the acceleration that leads to a change of direction. These two terms are important in the way that motion is perceived, so it is important to see how they can be computed. Since there are many different approaches to finding the quantities in this formula, it is useful to point out that calculations done earlier with numerical vectors give a way to organize the work efficiently. One needs only connect the use of the word *component* here with the earlier use of that word.

The velocity vector $\mathbf{v}(t) = v(t)\mathbf{T}(t)$, so it defines the direction $\mathbf{T}(t)$. In the previous sense of the word, the component of $\mathbf{a}(t)$ in the direction of $\mathbf{v}(t)$ is what we call here "the tangential component of acceleration", so it will be equal to d^2s/dt^2 even if its computation does not appear to involve differentiation of $v(t) = ds/dt$. It you have this component, you also have the projection by multiplying by the vector $\mathbf{T}(t)$. From the whole vector and the tangential projection, you can find the projection on the principal normal and its

component. This leads to an *algorithm* for computing curvature that is not easily summarized in a formula, but may be simpler than the formulas of section 13.3.

Exercises. The instructions are to find the tangential and normal components of acceleration.

$$\mathbf{r}(t) = \left\langle 3t - t^3, 3t^2 \right\rangle \tag{29}$$

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle \tag{31}$$

The textbook leads you through the use of this approach to derive Kepler's observations about planetary motion from Newton's hypothesis about the nature of the gravitational force. While exam questions in this course will concentrate on more primitive topics, this example establishes the historical importance of this use of vectors.

Combining the law of motion with the law of gravitation gives

$$m\mathbf{a} = \mathbf{F} = -\frac{GMm}{r^3}\mathbf{r}$$

which shows that $\mathbf{a} \parallel \mathbf{r}$. This shows that

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = \mathbf{v} \times \mathbf{v} + \mathbf{r} \times \mathbf{a} = 0,$$

which says that $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ is constant. In particular, $\mathbf{r}$ is confined to the plane perpendicular to $\mathbf{h}$. For each t we get coordinates in this plane by taking a unit vector $\mathbf{u}(t)$ in the direction of $\mathbf{r}(t)$, and $\mathbf{u}'(t)$ which is perpendicular to $\mathbf{u}(t)$ since the length of $\mathbf{u}(t)$ is constant. Direct computation shows that

$$\mathbf{h} = r^2 \mathbf{u} \times \mathbf{u}'.$$

Then,

$$\mathbf{a} \times \mathbf{h} = -GM\mathbf{u} \times (\mathbf{u} \times \mathbf{u}') = GM\mathbf{u}'.$$

Integrating gives

$$\mathbf{v} \times \mathbf{h} = GM\mathbf{u} + \mathbf{c}.$$

Finally, using

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h}$$

we get an equation in polar coordinates r, the distance from the sun, and θ, the angle from the vector $\mathbf{c}$, that are recognized as the equations of a conic.

The tools for deriving the connection between time and the area swept out by $\mathbf{r}$ are presented in an “Appled Project” on page 866.

This argument is not easy at this stage of our knowledge. I hope to return to the topic when more tools are available.