Definition of limit. It is useful at this point to review the definition of limit, extended to cover functions of several variables. For simplicity, we use only two variables.

We say that

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \qquad (*)$$

if we can force $f(x,y)$ to be arbitrarily close to L (within an ϵ given by *el Exigente*) by demanding that (x,y) be sufficiently close to (a,b) (at a distance no more than a δ that we choose), independent of direction.

Note that it is the whole sentence $(*)$ that is defined. The use of the equal sign in that definition is justified by . . . .

The main theorem. Limits are unique. That is, given the function f and the point (a,b), there is *at most one* choice of L satisfying $(*)$.

Proof. If not, let L_0 and L_1 be two different numbers that both satisfy $(*)$. The triangle inequality gives

$$|L_0 - L_1| \le |L_0 - f(x,y)| + |f(x,y) - L_1| \quad (\dagger)$$

for any point (x,y) in the domain of f. If you persist in claiming that L_0 and L_1 both satisfy $(*)$, I calculate

$$d = |L_0 - L_1|\,.$$

To say that $L_0 \neq L_1$ is to say that $d > 0$ (strictly). Then I choose $\epsilon = d/2$. Then, if (x,y) is any point that you think is close enough to (a,b), then both terms on the right side of $(\dagger)$ are less than $d/2$, which contradicts $(\dagger)$.

Non-existence of limits. The idea of this proof is used to identify cases in which limits do not exist. If there are two different values v_0 and v_1 such that $f(x,y) = v_0$ for some points (x,y) arbitrarily close to (a,b) while $f(x,y) = v_1$ for other points arbitrarily close to (a,b), then it is not possible for $f(x,y)$ at *all* points (x,y) close to (a,b) to be within $|v_0 - v_1|/2$ of any single value.

Continuity. Almost every function that we know how to express has the property that, at points where the expression for the function can be evaluated, that value may be used for L in $(*)$. This property is called **continuity**. Continuity means that that the value of a function at a point can be approximated by evaluating the function at a nearby point. This is what we do every time we trust our calculators to do much more than verify that $1+1=2$, so continuity is an abstraction of the idea of *computable*. The rules for building functions that we use are easily seen to preserve continuity, although complicated functions (for which δ is small) may require extraordinary care to give reasonable accuracy.

Exercises 14.1

7 For $f(x,y) = e^{x^2-y}$, find the domain and range of f and the value of $f(2,4)$.

9 For $f(x,y,z) = x^2 \ln(x-y+z)$, find the domain and range of f and the value of $f(3,6,4)$.

Exercises 14.2

Investigate limits of the following expressions at any point at which the expressions is not obviously continuous.

$$\frac{x^2}{x^2+y^2} \qquad (7)$$

$$\frac{xy}{\sqrt{x^2+y^2}} \qquad (11)$$

$$e^{-xy}\sin \pi z/2 \qquad (17)$$

$$\frac{x^2y^3}{2x^2+y^2} \qquad (35)$$