Motivation from geometry and calculation. The tangent line to the graph of a function was central to many of the applications of single variable calculus. One way to express th property of the tangent is Taylor's formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(\xi)}{2}(x - x_0)^2.$$

If x_0 is a number, an equation saying that y equals the sum of the first two terms on the right is the equation of a line. The last term is an **error term** giving the difference between the value $f(x)$ on the given curve and the y coordinate of the point on the tangent line for the same x. The ξ in this formula is a value between x_0 and x whose *existence* is asserted by Taylor's theorem although no attempt is made to find it. Instead, one uses its rough location to argue that $\left|f''(\xi)\right|$ is not too large. When $|x - x_0|$ is small, this error term is not just the smallest term in the expression, but **much** smaller than the other terms. This says that the function may be reasonably well approximated by the tangent line in some interval around x_0. The tangent lines of space curves met in Chapter 13 have

similar properties although proofs look a little different because space curves, including lines in space are defined parametrically.

The definition of the tangent plane will require that one plane approximates the surface near a point uniformly in all directions. Some surfaces that are otherwise well-behaved, like the cone $z^2 = x^2 + y^2$ fail to have such a tangent plane at the origin. Although it has many planes with some of the properties of a tangent, it is better not to try to weaken the definition to allow more tangent planes. The strict requirement has so many useful consequences, and is satisfied in many cases, that little is lost be leaving a few examples out of the theory.

Derivatives and tangent planes. The geometric version of the existence of a derivative of a function f at a point (a, b) is the existence of a tangent plane to the surface $z = f(x, y)$ at the point where $x = a$ and $y = b$. A tangent plane has an equation of the form $z = Ax + By + C$ for constants A, B and C, which we abbreviate $z = L(x, y)$ — L standing for **linear**. The definition giving the most efficient characterization of tangent planes is to require that, for all $\epsilon > 0$,

$$|f(x, y) - L(x, y)| < \epsilon\sqrt{(x-a)^2 + (y-b)^2}$$

for all (x, y) sufficiently close to (a, b), independent of direction. The expression on the right is chosen so that given any nonzero linear expression L, it must fail to bound L close to (a, b) for some ϵ. This allows the proof of the main theorem to be modified to show that there is at most one such L.

Using tangent lines to find tangent planes. Tangent lines to curves could have been defined in the same way. Hence, intersecting with the plane $y = b$ gives a curve whose equation is $z = f(x, b)$ and a tangent line to this curve at the point where $x = a$. Since tangent lines could have been characterized by the same ϵ-δ definition as tangent planes, these tangent lines *must* lie in the tangent plane. Finally, writing the equation of the tangent plane in the form

$$z = L(x, y) = A(x-a) + B(y-b) + C$$

we can use what we know about tangent lines to show that $C = f(a, b)$, A is the derivative of $f(x, b)$ with

respect to x evaluated at $x = a$, and B is the derivative of $f(a, y)$ with respect to y evaluated at $y = b$.

Finally, some calculus. The coefficients in the equation of the tangent line have been identified as values of derivatives, so finding them leads us, after a long detour into theory, to finding derivatives. The point of the theory has been to show the importance of the operation of differentiating with respect to one variable while holding other variables constant. This is easy to do: you simply follow the rules from single-variable calculus. It only remains to describe the notation. If z has been defined by some expression in the variables x and y, then the derivative of z with respect to x, treating y as a constant, is denoted

$$\frac{\partial z}{\partial x} \text{ or } D_x z.$$

As usual, the result of finding this derivative is an expression that usually involves x and y. To use this result to find a tangent line, you will need to evaluate the expression at $(x, y) = (a, b)$. There is no nice notation for this.

Alternatively, you can think of the given expression as defining z as a function of x and y, which you typically write $z = f(x, y)$. This notation means that the function f requires two input variables and the value of the function is found by using the given expression with the first variable assigned to x and the second to y. One of the things that is typically done with functions is to evaluate them at arbitrary expressions. Thus, if

$$f(x, y) = x + y^2,$$

then

$$f(y, x) = y + x^2.$$

What this example shows is that one should never assume that the variables used to describe a function have any significance whatsoever. Unfortunately, most of the other notations used in the textbook violate this rule. The only notation that doesn't is one that uses f_1 to stand for the derivative of the function f with respect to its first variable. This has the advantage that one can write $f_1(a, b)$ for the result of evaluating this function at the point (a, b), i.e., the result of

first differentiating the function and then evaluation the result at the base point.

Higher derivatives. Once one has a derivative, either as an expression or as a function, one can think of differentiating *that*. If one expects to do a lot of that sort of thing, an abbreviation is needed. Thus one writes

$$\frac{\partial^2 z}{\partial x^2} \text{ for } \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right); \; \frac{\partial^2 z}{\partial y \partial x} \text{ for } \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right);$$

and f_{12} for $(f_1)_2$.

Fortunately, you only need to be careful about the order of the variables in this expression when making the definition, since $f_{12} = f_{21}$ for functions that you will meet in practice.

Exercises 14.4 Find equation of tangent plane at indicated point.

$$z = y^2 - x^2 \qquad (-4, 5, 9) \tag{1}$$

$$z = \sqrt{4 - x^2 - 2y^2} \qquad (1, -1, 1) \tag{3}$$

$$z = \ln(2x + y) \qquad (-1, 3, 0) \tag{5}$$

$$z = e^x \cos(xy) \qquad (0, 0, 1) \tag{13}$$

Exercises 14.3 Find partial derivatives.

$$3x - 2y^4 \tag{11}$$

$$\frac{x - y}{x + y} \tag{15}$$

$$\sqrt{x^2 + y^2} \tag{33}$$