The chain rule.There are other derivatives involving functions of several variables that can be found. Suppose that z is given in terms of x and y, and that x and y are each given in terms of t. You could (and *Maple* does) use this to find z in terms of t and then calculate $D_t z$ (this is the neatest notation for this discussion).

Alternatively, you could apply the rules of differentiation to the expressions that you are given. Whatever the expression for z tells you is the last step in its computation is the first differentiation formula to be applied. In this process, expressions equal to $D_t z$ are obtained that can contain x, y, t, $D_t x$ and $D_t y$. Since x and y are given in terms of t, their expressions are used when you need to expand $D_t x$ and $D_t y$ in terms of t. The idea behind the chain rule for functions of several variables is to delay the expansion of $D_t x$ and $D_t y$ as long as possible. This gives an expression for $D_t z$ in which t has yet to appear outside of a subscript.

The fundamental linearity of differentiation formulas. Let's take a close look at the formulas of elementary calculus:

$$D_t(x+y) = D_t x + D_t y \qquad (S)$$

$$D_t(x \cdot y) = D_t x \cdot y + x \cdot D_t y \qquad (P)$$

$$D_t\big(f(x)\big) = f'(x) \cdot D_t x \qquad (C)$$

These rules suffice to differentiate all the functions met so far, when supplemented by special formulas for differentiating functions given by $f(x) =$ one the following expressions: a constant, x^n, e^x, $\ln x$, $\sin x$, or $\cos x$, $\arctan x$. A few more formulas are obtained to avoid deriving them from other formulas every time they are needed, but this short list of formulas is a good summary of elementary calculus. Of course, the course really deals with the understanding of functions that allows you to use these formulas and apply the results to things in the real world that can be modeled by this mathematical abstraction.

The thing to notice about formulas (S), (P) and (C) is that each *term* contains a *factor* that is a single application of D_t. This means that, in the setting at the start of this lecture,

$$D_t z = A \cdot D_t x + B \cdot D_t y, \qquad (*)$$

where A and B are expressions involving x and y.

Connection with partial derivatives. Formula $(*)$ holds independent of the dependence of x and y on t. The special cases used to define partial derivatives (or, at least, their values at particular points) are obtained by using the parameterizations: (1) $x = t$, $y = b$; or (2) $x = a$, $y = t$. This shows that $A = D_x z$ and $B = D_y z$. The usual statements of the chain rule are obtained by translating this statement into different notations.

A word about proving the chain rule. This discussion has emphasized how the chain rule appears in calculations, suppressing all aspects of proof. However, this is not completely lacking in mathematical rigor. The rules of elementary calculus were obtained with proofs based on a definitions in terms of limits. In fact, it would have been better to use the "good linear approximation" version introduced in connection with our discussion of the tangent plane. Inside these proofs are rules for obtaining a δ from a given ϵ, although we usually don't look that closely.

For functions given by expressions that we recognize, our calculation is a proof that the derivatives exist. In fact, a close examination of what we have when we have reached the stage of formula $(*)$ shows that the surface defined by the given expression of z in terms of x and y has a tangent plane wherever the calculation is valid.

What this approach does *not* do is tell how to deal with a new function of several variables. What should be done, if such a function is ever met, is to show from the definition of the function that it has a tangent plane, and then prove a version of the chain rule we have already stated in the case where the graph of $z = f(x, y)$ has a tangent plane and the curve $x = g(t)$, $y = h(t)$ is differentiable.

Since we are not likely to meet any new functions of several variables in this course, this approach is more suitable for *Advanced Calculus*. However, in a sense, this was exactly what was done in deriving formulas (S) and (P)! Look closely at the expressions $x + y$ and $x \cdot y$. What are the partial derivatives? Does this agree with the statement given for the chain rule? (Of course, these questions are rhetorical.)

Exercises

Find derivatives and identify role of the chain rule. In these examples, each variable plays a definite role. If desired, everything could be expressed in terms of the independent variable(s) only. This illustrates the way that computations are organized in a spreadsheet.

1. $z = x^2y + xy^2$, $x = 2 + t^4$, $y = 1 - t^3$.

3. $z = \sin x \cos y$, $x = \pi t$, $y = \sqrt{t}$.

7. $z = x^2 + xy + y^2$, $x = s + t$, $y = st$.

9. $z = \arctan(2x + y)$, $x = s^2t$, $y = s \ln t$.

19. $w = x^2 + y^2 + z^2$, $x = st$, $y = s \cos t$, $z = s \sin t$ (at $s = 1$, $t = 0$).

Implicit differentiation works as in the single variable case. In addition, there is an expression for the result in terms of partial derivatives. This is probably not as useful as it appears to be.

25. $x^2 - xy + y^3 = 8$; find dy/dx.

29. $xy^2 + yz^2 + zx^2 = 3$; find $\partial z/\partial x$ and $\partial z/\partial y$.