Gradients. There is one more thing to be seen in $(*)$. Whenever one has a sum of terms, each of which is a product of something of one type and something of another, it should be viewed as a dot product of vectors. We have already met the vector $\langle D_t x,\, D_t y\rangle$ as the velocity vector when $\mathbf{r}(t) = \langle x,\, y\rangle$ gives the position of a point at time t. This model suggests that it won't be long before we try to write $\mathbf{r}'(t)$ in the form $(ds/dt)\mathbf{T}$, but first we collect the other factors into a vector $\big\langle D_x z,\, D_y z\big\rangle$. When differentiating a function f instead of an expression z, this has the form $\langle f_1,\ f_2\rangle$. In this form, it is easy to imagine the generalization to functions of any number of variables. This vector is called the **gradient** of f and denoted ∇f. Gradients are very much a "function thing" since it emphasizes the domain of the function rather than the range — there is no good notation for the same object constructed from an expression.

Like all other derivatives, gradients will be evaluated at points of their domain when they appear in applications.

Directional derivatives. The chain rule can now be expressed as

$$D_t f\big(\mathbf{r}(t)\big) = \nabla f\big(\mathbf{r}(t)\big)\cdot \mathbf{r}'(t).$$

While f is an ordinary real valued function on some $\mathbb{R}^d$ (with $d = 2$ or $d = 3$ for most of the examples in this course, but a common description of those cases leads immediately to a vast generalization), ∇f is something else: for each point $P \in \mathbb{R}^d$, $\nabla f(P)$ is a vector in $\mathbb{R}^d$ based at P. One term frequently used for such a function $\mathbb{R}^d \to \mathbb{R}^d$ is **vector field**. The appearance of $\nabla f(P)$ in an inner product suggests that its principal interpretation will be in terms of expressions of the form $\nabla f(P)\cdot \mathbf{v}$. In particular, if $\mathbf{v}$ is a unit vector, this expression is called the **directional derivative** of f in the direction $\mathbf{v}$. The special case in which $\mathbf{v}$ lies along one of the coordinate axes has already been met under the name "partial derivative".

Now, as promised, we write $\mathbf{r}'(t) = (ds/dt)\mathbf{T}$ to express $D_t f\big(\mathbf{r}(t)\big)$ as the product of the *speed* with which $\mathbf{r}(t)$ is drawing the curve and the directional derivative of f in the *tangential direction* of the curve.

There are two important special cases: $\mathbf{v} \perp \nabla f$ is equivalent to the directional derivative being zero; the directional derivative takes its maximal value when $\mathbf{v} \parallel \nabla f$. Since

$$\nabla f \cdot (-\mathbf{v}) = -\nabla f \cdot \mathbf{v},$$

the minimum value of the directional derivative is in the *anti-parallel* direction.

An important application is that level curves of functions on $\mathbb{R}^2$, or level surfaces in $\mathbb{R}^3$, are perpendicular to the gradient of the function.

Changing coordinates. Since the last step of computing a gradient of f is to draw that vector field at points of the domain of f, the coordinates used in its computation have been pushed into the background. This suggests that it should be possible to perform these computations in other coordinate systems and get the same geometric answer. This is indeed true, with one important requirement — since inner products play an important role in the theory, only systems in which the coordinates are the components

of a set of *mutually perpendicular unit vectors* can be used. While it is usually better to use a disjoint set of names for the elements of different coordinate systems, the restriction to orthonormal coordinates allows the same name to appear in different coordinate systems as long as it means the component with respect to the same vector.

Exercises. Find gradient ∇f in general and at point P. If a direction at P is given, find the directional derivative in that direction.

#3. $f(x, y) = x^2y^3 + 2x^4y$, $P(1, -2)$, angle $\pi/3$ from positive x-axis.

#9. $f(x, y, z) = xy^2z^3$, $P(1, -2, 1)$, in direction $(1/\sqrt{3})\langle 1, -1, 1\rangle$.

#15. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $P(1, 2, -2)$, direction of $\langle -6, 6, -3\rangle$.

More Exercises. Find tangent plane at given point.

#37. $x^2 + 2y^2 + 3z^2 = 21$, $P(4, -1, 1)$.

#39. $x^2 + y^2 - z^2 - 2xy + 4xz = 4$, $P(1, 0, 1)$.

#57. Both $z = x^2 + y^2$ and $4x^2 + y^2 + z^2 = 9$ at $P(-1, 1, 2)$. Then give tangent line to curve of intersection.

Implicit functions. In section 14.5, some exercises were devoted to finding partial derivatives of implicitly defined functions. Such exercises should be approached with caution, especially if you are tempted to use the formulas for the derivatives of such functions given in the textbook. The notation for partial derivatives often fails to specify the variables held constant in the differentiation — they are assumed to be all other variables. However, if the equations of a surface is used as an implicit definition of z as a function of x and y, then z has two different meanings in the same problem: it is one of three independent variables in the equation of the surface; it is also the depend variable for the implicitly defined function. The variables x and y are independent variables in both interpretations, but they are part of a triple of variables in the first interpretation, while they are the only independent variables in the second.

If you are given a point P on a surface defined by

$F(x, y, z) = 0$, the calculus of implicit functions shows how to find the value of the partial derivatives of any function giving z in terms of the remaining variables at P whenever $F_z(P) \neq 0$. The implicit function theorem says (subject to some technical hypotheses) that such a function exists in a neighborhood of P. There may be no useful expression for the function, but the concept of function used in the underlying theory of the calculus is found to be general enough to include solutions of such equations.

The study of tangent planes interprets the condition the $F_z(P) \neq 0$ as saying that there is a tangent plane at P and that plane is not perpendicular to the xy plane. Such a tangent plane has an equation that can be solved for z, and that solution approximates the dependence of z on the other variables on the surface $F(x, y, z) = 0$ near P. The implicit function theorem says that this approximation can be refined to a function that expresses z exactly. It may also be possible to solve $F(x, y, z) = 0$ for x or y.