

Overview of beginning of chapter 14. If $f(x, y, z)$ is a real-valued function on \mathbb{R}^3 , its **gradient** ∇f was introduced in Section 14.6. When it was introduced, we noted that it was an new kind of function called a **vector field**. In Section 16.1, a definition of this object finally appears, with some examples from physics. It is quickly noted that a vector field \mathbf{F} that is of the form ∇f is very special. The word **conservative** is introduced to describe such vector fields, and f is given the special designation of the **potential function** of \mathbf{F} . This is supposed to inspire a feeling of warm fuzzies among those who have met these terms in a physics course. It is motivated by the concept of *potential energy*, which is an invention that allows one to claim a law of *Conservation of energy*. The change in the more observable *kinetic energy* is given by the **work integral**, which is the integral of the *tangential component* with respect of arc length of the force acting on an object. Formula (7) in Section 11.9 noted that only the tangential component of acceleration contributed to changes of speed, and this is what is being

measured here. This work integral takes the form

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds.$$

The \mathbf{T} and ds in this formula should not be taken seriously, they only serve to establish the link with physics. The \mathcal{C} in this notation represents the curve over which the object moves, and if \mathcal{C} is given by a vector function $\mathbf{r}(t)$, then

$$\mathbf{T} ds = \mathbf{r}'(t) dt$$

from the definitions of \mathbf{T} and s . The chain rule of elementary calculus shows that a change of parameter gives the usual change of variable in this integral, and hence, does not change the value of the integral. Thus, although we **describe** the integral using $\mathbf{T} ds$, we **compute** it using $\mathbf{r}'(t) dt$.

If $\mathbf{F} = \nabla f$, then the chain rule for functions of several variables shows that this integral is equal to the difference of the values of f at the ends of the curve \mathcal{C} . This is the first generalization of the fundamental theorem to appear in this course.

Integrals of vector fields. The text mentions some integrals of scalar functions with respect to arc length along curves. Such integrals (except for the calculation of arc length itself) are artificial. Only integrals of vector fields like the *work integral*, seem to appear in applications, and only these integrals have interesting mathematical properties. If \mathcal{C} is a curve parameterized by $\mathbf{r}(t)$ for $a \leq t \leq b$, then we describe this integral as

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T} ds$$

or

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r},$$

and evaluate it as

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt,$$

which we think of as a substitution in the previous expression. The value of this expression is independent of parameterization. If $\mathbf{F} = \langle P, Q, R \rangle$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r},$$

can be expanded as

$$\int_{\mathcal{C}} P dx + Q dy + R dz. \quad (*)$$

In these formulas, P , Q , and R are functions of x , y , and z . Expanding everything in terms of the parameter t gives the same formula as before. This shows the power of the notation of expressions when doing calculus. Indeed, some of the things that you think you see in the form $(*)$ turn out to really be there and allow simplifications. For example, if you have $\int x dx$, it is equal to the difference of the values of $x^2/2$ at the two ends of the path.

If you need to give a piecewise definition of the curve \mathcal{C} , then the resulting integral will normally be written as the sum of the integrals over the pieces. For example, to integrate \mathbf{F} in the counterclockwise direction around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$, you form

$$\int_0^1 \mathbf{F}(x, 0) dx + \int_0^1 \mathbf{F}(1, y) dy$$

$$+ \int_1^0 \mathbf{F}(x, 1) dx + \int_1^0 \mathbf{F}(0, y) dy.$$

The fundamental theorem. If $F = \nabla f$, then the integrand in (*) is

$$f_1(x, y, z) dx + f_2(x, y, z) dy + f_3(x, y, z) dz,$$

which the chain rule for several variables says is just $df(\mathbf{r}(t))$. The fundamental theorem of single variable calculus says that this integrates to the difference of the values of f at the points $\mathbf{r}(t)$ obtained from the endpoints of the interval in t over which you are integrating. These values of $\mathbf{r}(t)$ are just the endpoints of the arc \mathcal{C} . In particular, $f(x, y, z)$ can be determined up to an additive constant by integrating \mathbf{F} along any path from a fixed base point (x_0, y_0, z_0) to (x, y, z) .

Independence of path. The result just mentioned shows that the integral of a conservative vector field depends only on the endpoints of the arc \mathcal{C} and not on the details of how \mathcal{C} gets from one of those points to the other. However, you should remember to verify

the hypothesis of this theorem before jumping to its conclusion. This is mostly a theoretical result, not a shortcut for evaluating integrals. Direct use of the definition is an important skill which should not be abandoned before it is mastered.

Finding potential functions. Given a vector field \mathbf{F} that is defined everywhere, we can build a path from (x_0, y_0, z_0) to (x, y, z) by following lines parallel to the axes to (x, y_0, z_0) and (x, y, z_0) . If $\mathbf{F} = \nabla f$, the integral along this path gives a possible value of f , and all other choices of f differ from this by our old friend “ $+ C$ ”.

A necessary condition for a field to be conservative. Clairaut’s theorem tells that $f_{ij} = f_{ji}$. If $\mathbf{F} = \nabla f$, this says that the partial derivative of the i -th component of \mathbf{F} with respect to the j -th variable is everywhere equal to the partial derivative of the j -th component of \mathbf{F} with respect to the i -th variable.

If \mathbf{F} does not have this property, it cannot be conservative. Conversely, if it has this property, the standard path can be used to find a tentative choice of f . One can then compare the gradient of this function to \mathbf{F} .

Usually, one finds that $\nabla f = \mathbf{F}$. Green's theorem can be used to formulate and prove a precise statement.

The key example. If

$$\mathbf{F}(x, y) = \left\langle \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right\rangle,$$

then it appears that our necessary condition is satisfied. However, \mathbf{F} and its derivatives fail to be defined at $(0, 0)$. This causes \mathbf{F} to fail to be conservative. It is easy to compute its integral around the unit circle, $x = \cos t$, $y = \sin t$, from $t = 0$ to $t = 2\pi$.

This simple example shows that independence of path requires that it be possible to compute the integral along a family of paths that describe how one can deform one path into another. A single point that prevents integrals through it from being defined can cause independence of path to fail.

The value of the integral we have given on any path not containing $(0, 0)$ is always an integer multiple of 2π . That integer can be interpreted as the number of

times the path goes around the origin in the counter-clockwise sense (so that it is -1 if you go once around in the clockwise sense).

Exercises 16.2

Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

#19. $\mathbf{F} = \langle x^2 y^3, y\sqrt{x} \rangle,$

$$\mathbf{r} = \langle t^2, -t^3 \rangle,$$

$$0 \leq t \leq 1.$$

#21. $\mathbf{F} = \langle \sin x, \cos y, xz \rangle,$

$$\mathbf{r} = \langle t^3, -t^2, t \rangle,$$

$$0 \leq t \leq 1.$$

Exercises 16.3

Is $\mathbf{F} = \nabla f$?

#3. $\mathbf{F} = \langle 6x + 5y, 5x + 4y \rangle$

#5. $\mathbf{F} = \langle xe^y, ye^x \rangle.$