Definition of double integral. The definition of the integral in single variable calculus in terms of Riemann sums serves several purposes.

From the point of view of pure mathematics, it leads to a theorem that asserts that every continuous function can be integrated.

From the point of view of numerical analysis, the definition indicates that the integral can be approximated by a finite sum.

From the point of view of physics, the integral is seen to provide a measurement for ordinary objects that agrees with what would be obtained if the body were approximated by a large number of point masses.

In order to have all of these viewpoints available for multiple integrals, a double Riemann sum of a function $f(x, y)$ over the rectangle

$$\mathcal{R} = \left\{ x, y \in \mathbb{R}^2 : a \le x \le b, c \le y \le d \right\}$$

is defined by partitioning $\mathcal{R}$ into rectangles of small diameter. The integral is allowed to exist if all of

the sums give almost the same answer when the little rectangles are forced to have small diameters.

Rectangles are dull. In the one variable case, the fact that continuous functions are integrable suffices for many examples, and is easily extended to show that **piecewise** continuous functions are integrable. In the several variable case, even such a simple problem as finding the area of a circle requires that the function that is 1 inside the circle and 0 outside be integrable. The key to including this in the theory is that boundary can be covered by rectangles of arbitrarily small total area. This is only a technical device allowing a special definition to be extended to include more useful cases. It has very little connection with the way in which we do calculus with multiple integrals.

For doing calculus, the basic regions are those for which the intersection with a vertical line is either empty or a single interval. The endpoints of this interval will depend on x. The text calls this “Type I” and writes the description as

$$\left\{ (x, y) : a \le x \le b, g_1(x) \le y \le g_2(x) \right\}.$$

Some examples are: (1) the unit circle with center $(0, 0)$,

$$a = -1, b = 1,$$
$$g_1(x) = -\sqrt{1 - x^2}, g_2 = \sqrt{1 - x^2};$$

(2) the triangle with vertices $(0, 0)$, $(x_0, 0)$, $(0, y_0)$,

$$a = 0, b = x_0, g_1(x) = 0, g_2(x) = y_0\left(1 - \frac{x}{x_0}\right);$$

(3) the area below the line $y = x + 2$ and above the parabola $y = x^2$,

$$a = -1, b = 2, g_1(x) = x^2, g_2(x) = x + 2.$$

Example (3) should be examined more closely. The stated description identifies the functions $g_1(x)$ and $g_2(x)$, but the values of a and b need to be calculated by solving the inequality $g_1(x) \leq g_2(x)$. Any weakness in algebra can often be overcome by sketching the given curves and using the sketch to formulate an

algebraic problem that you can solve to complete the description.

Reversing the roles of x and y leads to a region of "Type II". Still more general regions can be obtained as unions of regions of these type that intersect only along curves.

Iterated integrals. Once a description of a Type I region has been found, the integral of a function $f(x, y)$ over the region is given by

$$\int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right) dx.$$

The inner integral, with respect to y is formed treating x as a constant. For each x this is a definite integral, so all dependence on y will disappear when this is evaluated. The resulting expression may still depend on x, but when it is integrated with respect to x between constant limits, the answer will be a real number.

In writing such integrals, it is customary not to write the parentheses, but the limits on the first integral sign are associated with variable in the last differential factor. Such a pair serves to delimit the expression being integrated. This variable is also available as a parameter in describing limits of integration and the integrand of any included integral.

Exercises 15.4 Integrate $f(x, y)$ over $\mathcal{D}$.

#7. $f(x, y) = x^3y^2$,

$$\mathcal{D} = \{ (x, y) : 0 \leq x \leq 2, -x \leq y \leq x \}.$$

#9. $f(x, y) = 2y/(x^2 + 1)$,

$$\mathcal{D} = \left\{ (x, y) : 0 \leq x \leq 1, 0 \leq y \leq \sqrt{x} \right\}.$$

#15. $f(x, y) = y^3$, $\mathcal{D}$ is the triangle with vertices $(0, 2)$, $(1, 1)$, $(3, 2)$.

#17. $f(x, y) = 2x - y$, $\mathcal{D}$ is the circle of radius 2 with center at the origin.

#19. $f(x, y) = x^2 + y^2$, $\mathcal{D}$ bounded by $y = x^2$ and $x = y^2$ (described as finding a volume).

#25. $f(x, y) = 1 - x - y$, $\mathcal{D}$ bounded by $x = 0$, $y = 0$, $1 - x - y = 0$ (described as finding a volume).

#39. $f(x, y) = e^{x^2}$,

$$\mathcal{D} = \{ (x, y) : 0 \leq y \leq 1, 3y \leq x \leq 3 \}$$

(The iterated integral with this Type II description was given, but the Type I description was to be used in calculation).

#51. $f(x, y) = x^2 \tan x + y^3 + 4$,

$$\mathcal{D} = \left\{ (x, y) : x^2 + y^2 \leq 2 \right\}$$

(exploiting symmetry with respect to both axes).