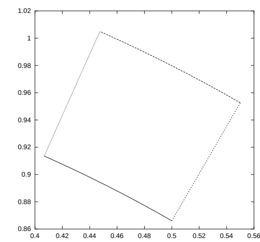
Polar coordinates. The equations $x = r\cos\theta$ and $y = r\sin\theta$ relate the rectangular coordinates x and y to the polar coordinates r and θ. The periodicity of the trigonometric functions shows that adding 2π to θ and using the same r gives the same values of x and y. A slightly closer look at the trig functions shows that adding π to θ and multiplying r by (-1) also gives the same x and y. This operation shows that the circle represented by the polar equation $r = \cos\theta$ is completely traversed when θ goes through an interval of length π. Since points in the xy-plane have infinitely many representations in polar coordinates, it will be important when constructing integrals to know how to produce inequalities in r and θ that represent **only one** set of coordinates for the points in a given region. This will require some care, but it is not particularly difficult if you allow yourself to be guided by a picture.

There are a small number of curves that give the common examples of curves given by polar equations. You should do enough exercise to meet them all.

Polar rectangles. A set given by the conditions $a \le r \le b$ and $\alpha \le \theta \le \beta$ is called a **polar rectangle**. Here is an example.



The definition of the double integral can be copied for polar coordinates by partitioning a portion of the plane into small polar rectangles, i.e., polar rectangles in which both $b - a$ and $\beta - \alpha$ are small. Although slightly bent, the sides of a polar rectangle are seen to have lengths Δr and $r\,\Delta\theta$ and an area $r\,\Delta r\,\Delta\theta$. In fact, a polar rectangle can be seen to have **exactly** this area with $r = (a+b)/2$.

The double integral of the function $f(x, y)$ over a region $\mathcal{R}$ in the xy-plane was defined as a limit of sums over small rectangles of the value of f somewhere in

the integral times the area of the rectangle. This is often interpreted as the volume of a figure with base $\mathcal{R}$ whose height at (x, y) is $f(x, y)$.

The only expression in polar coordinates that has a chance of representing the same thing is a sum over polar rectangles of $f(r\cos\theta, r\sin\theta)$ for values of r and θ within the intervals specifying the polar rectangle times **the area of the polar rectangle**, which we know to be $r(\Delta r)(\Delta\theta)$ for a value of r somewhere inside the polar rectangle.

This approach to integration in polar coordinates is based on duplicating the definition of integrals using Riemann sums. If you don't believe that Riemann sums are useful, this isn't completely satisfactory. In later sections, we will see other ways to explain the equation

$$dA = dx\,dy = r\,dr\,d\theta$$

gives polar integrals that agree with the integrals defined in rectangular coordinates. However, the underlying geometry makes it clear that this is the only computation that has a chance of giving the right answer.

If we want to calculate

$$\iint_{\mathcal{R}} f(x, y)\,dx\,dy$$

using polar coordinates, we must form

$$\iint_{\mathcal{R}^*} f(r\cos\theta, r\sin\theta)\,r\,dr\,r\theta,$$

where $\mathcal{R}^*$ stands for the description in polar coordinates of $\mathcal{R}$. In exercises in this course, the rectangular description $\mathcal{R}$ and the polar description $\mathcal{R}^*$ are usually found using a picture of $\mathcal{R}$ which is interpreted in polar coordinates. The picture is also used to assure that everything has been represented so that $dx\,dy$ and $r\,dr\,d\theta$ represent positive fragments of area.

In contrast to rectangular coordinates where "Type I" and "Type II" descriptions must often both be studied before deciding how to calculate an integral, polar integrals are almost always calculated by integrating first with respect to r. This is due to polar coordinates being used primarily to describe curves in which r is an explicit function of θ.

To find the area inside a polar curve $r = f(\theta)$ using double integrals, we form

$$\int_{\theta_0}^{\theta_1} \int_0^{f(\theta)} r\,dr\,d\theta = \int_{\theta_0}^{\theta_1} \frac{1}{2} f(\theta)^2\,d\theta$$

in agreement with the formula introduced in Section 10.5.

Since the double integral gives the previous formula so easily using one order of integration, there is no reason not to formulate *all* problems as multiple integrals instead of relying on the earlier formulas for getting single integrals. Multiple integrals always have the advantage of allowing additional methods of evaluation which may lead to simpler calculation.

Exercises. Find the integral

$$\iint_{\mathcal{R}} f(x, y)\,dA.$$

#7 $f(x, y) = x$, $\mathcal{R}$ is the disk of radius 5 centered at the origin.

#9 $f(x, y) = xy$, $\mathcal{R}$ is the region in the first quadrant between $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.

#11 $f(x, y) = e^{-x^2-y^2}$, $\mathcal{R}$ is the semicircle bounded by $x = \sqrt{4 - y^2}$ and the y-axis.

#13 $f(x, y) = x^2 + y^2$, $\mathcal{R}$ is the region bounded by the spirals $r = \theta$ and $r = 2\theta$ for $0 \leq \theta \leq 2\pi$

#15 $f(x, y) = 1$ (finding area), $\mathcal{R}$ is one loop of the rose $r = \cos 3\theta$.

#23 Find volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

#27 Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$$

by converting to polar coordinates.