Simple regions. We are now ready to give a form of the fundamental theorem of calculus relating double integrals over regions in the plane with line integrals around the boundary of the region. There is an annoying complication that arises when trying to state such a theorem: double integrals were described in an unoriented form useful for finding areas, moments and other physical quantities, but line integrals were given with an orientation that was a natural consequence of the use of a parameterization to reduce their evaluation to the familiar one-dimensional definite integral.

Setting up a double integral usually required a picture to check that all parts of the region were counted exactly once. The regions that we met were usually those described as "Type I", with a description of the form

$$\mathcal{R} = \left\{ (x, y) : a \le x \le b, g_1(x) \le y \le g_2(x) \right\}.$$

This says that a vertical line that meets $\mathcal{R}$, meets it in an interval, and the vertical line $x = c$ meets $\mathcal{R}$ only for an interval of values of c. The formalism serves only to provide names to the endpoints of these intervals.

Integrals over more complicated regions can be found by writing the region as a union of Type I regions that intersect only in line segments, so that the integral over the whole will be the sum of integrals over the pieces.

Similarly, there were regions of "Type II" in which horizontal lines met the region in intervals.

A simple region is something that is of both Type I and Type II. A typical example is the portion of the first quadrant under the graph of a decreasing function. A common problem involving simple regions is to be given an integral using the Type I description of the region that looks difficult to evaluate, and to convert it to the equivalent integral using the Type II description of the region. This equivalent form turns out to be easy to evaluate, so you do it.

Proof of Green's theorem. Although we have not yet stated the theorem, we have all that we need to prove it.

The evaluation of the integral of $f(x, y)$ over our region $\mathcal{R}$ of Type I as an iterated integral proceeds by finding a function $F(x, y)$ with $F_2 = f$ so that

$$\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy = F\big(x, g_2(x)\big) - F\big(x, g_1(x)\big).$$

Then

$$\iint_{\mathcal{R}} f(x, y)\, dx\, dy = \int_a^b F\big(x, g_2(x)\big)\, dx - \int_a^b F\big(x, g_1(x)\big)\, dx.$$

To get our proof, we note that the integrals on the right are of the form

$$\int_{\mathcal{C}} F(x, y)\, dx,$$

where $\mathcal{C}$ is a curve forming part of the boundary of $\mathcal{R}$. The boundary of $\mathcal{R}$ may also include line segments lying along $x = a$ or $x = b$, but these have no contribution to an integral with a dx factor. We may

insert a multiple of such integrals to get a geometrically satisfying result without changing the value of the integral.

Similar integrals with respect to y are found by using a Type II description to integrate $f(x, y)$ with respect to x.

We need only interpret these results to get Green's theorem.

The boundary of a region. Consider the case in which $\mathcal{R}$ is the unit circular disk defined by

$$\Big\{ (x, y) : x^2 + y^2 \leq 1 \Big\}.$$

The boundary is the circle with equation $x^2 + y^2 = 1$, which we always parameterize by $x = \cos t$ and $y = \sin t$. In this parameterization, we go around the circle in a counterclockwise direction, and the disk is always to our left.

Something similar can be done for every simple region. To keep the region to our left, we should go from $x = a$ to $x = b$ along the bottom of the region, then up along $x = b$, then from $x = b$ to $x = a$ along the top, and finally down along $x = a$.

Note that, if a Type I region is cut along the graph of a function, that curve would be traversed from left to right as the bottom of the upper part of the region and from right to left as the top of the lower part of the region. In this way, we see that a line integral over the boundary of the union of the two pieces is the sum of line integrals over the boundaries of the two pieces.

If $\mathcal{C}$ is a closed curve, bounding a region $\mathcal{D}$, we use

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

to denote an integral that goes once around $\mathcal{C}$ in the direction that keeps $\mathcal{D}$ to the left. For simple regions, this direction on $\mathcal{C}$ is easy to find. This construction extends to regions that can be cut into simple parts. At some point, one should show that the oriented boundary of a region does not depend on how the region is cut into simple pieces. The regions for which this gives a well-defined oriented boundary exhaust all regions that you are likely to meet, although

the full mathematical generality allows regions that can be approximated by such regions.

A statement of Green's theorem. If $\mathcal{D}$ is a simple region with boundary $\mathcal{C}$, then

$$\oint_{\mathcal{C}} P(x, y)\,dx = \iint_{\mathcal{D}} -P_2(x, y)\,dx\,dy$$
$$\oint_{\mathcal{C}} Q(x, y)\,dy = \iint_{\mathcal{D}} Q_1(x, y)\,dx\,dy$$

Note that the choice of orientation of the boundary leads to the pattern of signs in these formulas.

Formula (1) in Section 16.4 results from adding these two formulas.

Conservative vector fields. The test for $\langle P, Q\rangle$ being conservative is $Q_1 - P_2 = 0$. Green's theorem thus includes the statement that the integral of a conservative vector field around the boundary of any region is zero. One needs these derivatives to exist inside the region bounded by the curve for this proof to work. We have already mentioned an example where the derivatives fail to exist at a single point and the integral is nonzero along closed curves enclosing that point.

The typical exercise in the use of Green's theorem involves evaluation of line integrals around closed curves by transforming them into double integrals. Of course, the proof of Green's theorem shows that the evaluation of the double integral involves replacing it by a line integral — but the resulting line integral is often much simpler than the one you started with. The two line integrals related by this operation differ by the integral of a conservative vector field.

The examples obtained by interchanging order of integration in simple regions also lead to line integrals of vector fields whose difference is conservative.

Not so simple regions. After performing detailed calculations on simple regions, the results can be applied to more general regions by formal means. The main tool cuts along a straight line. The integrals associated with the whole region are easily seen to be the sum of corresponding integrals for the parts.

The analysis of limits enters to show that certain line integrals along paths confined to sets of small diameter are bounded in terms of the area of the set. Such a bound is certainly necessary for Green's theorem to apply. As long as the estimates are good for small regions, the limit machinery will allow crude bounds to be refined.

Since line integrals are simpler from the theoretical point of view, they can be used to interpret what the corresponding double integrals must be for Green's theorem to remain true. This leads to an interpretation of oriented and multiple areas. Although this makes the double integral more difficult to interpret, it leads to simpler rules for manipulating them, that is to a *calculus*. The calculation of the integrals no longer requires a picture, although a picture may be needed to interpret the result. That is, if you go around $r = 1 + 2\cos\theta$ from $\theta = 0$ to $\theta = 2\pi$, the region inside the inner loop of the curve is counted twice, while the portion between the two loops is counted once.

Changes of variable. If we have functions giving P and Q in terms of x and y and x and y in terms of u

and v, and a closed curve $\mathcal{C}$ described by giving u and v (and hence also x and y) in terms of t, then

$$\int_{\mathcal{C}} P\,dx + Q\,dy$$

can be viewed as a line integral in the xy-plane, or in a uv-plane, and can be evaluated as an ordinary integral in terms of t. Formulating Green's theorem in the uv-plane gives a formula for allowing the uv coordinates to evaluate integrals formulated in terms of x and y. A general discussion of this formula appears in section 15.9 although the special case of polar coordinates has already been mentioned.

Deriving this formula from Green's theorem allows all of the mathematical difficulty to be confined to this one theorem. The argument given to justify the formula for polar coordinates appeared to depend on having simple descriptions of a region in both rectangular and polar coordinates. Of course, this will be true in all cases in which you would use this formula However, for regions not having such simple descriptions, the approach through Green's theorem assures us that the interpretation of all integrals is correct.

Exercises 16.4

Evaluate the integral $\oint_{\mathcal{C}} P\,dx + Q\,dy$ as many ways as you can.

#3. $P = xy$, $Q = x^2y^3$, $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$.

#7. $P = e^y$, $Q = 2xe^y$, $\mathcal{C}$ is the square with sides $x - 0$, $x = 1$, $y = 0$, $y = 1$.

#9. $P = y + e^{\sqrt{x}}$, $Q = 2x + \cos(y^2)$, $\mathcal{C}$ is the boundary of the region enclosed by $y = x^2$ and $x = y^2$.

#21. $P = -y/2$ $Q = x/2$ (area computation),

$$\mathcal{C} = \left\langle \cos^3 t, \sin^3 t \right\rangle (0 \le t \le 2\pi).$$

#21. Area of triangle with vertices $(0,0)$, (x_0, y_0), (x_1, y_1).