Areas in a plane. Suppose we have vectors

$$\mathbf{v}_0 = \langle a_0, b_0, c_0 \rangle \quad \mathbf{v}_1 = \langle a_1, b_1, c_1 \rangle$$

based at a point $P(x_0, y_0, z_0)$ in $\mathbb{R}^3$. Then the four points P, $P+\mathbf{v}_0$, $P+\mathbf{v}_1$, $P+\mathbf{v}_0+\mathbf{v}_1$ are the vertices of a parallelogram $\mathcal{P}$ in space, which lies in the plane

$$Ax + By + Cz = D$$

where

$$\langle A, B, C \rangle = \mathbf{v}_0 \times \mathbf{v}_1$$

and

$$D = Ax_0 + By_0 + Cz_0.$$

The area of $\mathcal{P}$ is

$$|\mathbf{v}_0 \times \mathbf{v}_1| = \sqrt{A^2 + B^2 + C^2}.$$

If we project this figure into the xy plane, we get a parallelogram with one vertex at $P_0(x_0, y_0, 0)$ and sides given by the vectors

$$\mathbf{w}_0 = \langle a_0, b_0, 0 \rangle \quad \mathbf{w}_1 = \langle a_1, b_1, 0 \rangle$$

whose area is

$$|\mathbf{w}_0 \times \mathbf{w}_1| = |\langle 0, 0, C \rangle| = |C|$$

The ratio of the area of the projection to the area of $\mathcal{P}$ is

$$\frac{|C|}{\sqrt{A^2 + B^2 + C^2}}.$$

Note that this quantity does not change if A, B, C, and D are all multiplied by a number λ to obtain a different equation of the same plane.

Also note that the projection has area zero if $C = 0$, which says that the equation of the plane does not depend on z, or that the plane is perpendicular to the xy plane.

On the other hand, if $C \neq 0$, we can invert this ration to find the amount that the area of the projection should be multiplied by to obtain the area of $\mathcal{P}$. This ratio has the form

$$\sqrt{\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 + 1},$$

so it is always at least 1. The quantities A/C and B/C in this formula are the negatives of the coefficients of x and y when the equation of the plane is solved for z.

Area on the graph of a function. The main principle in the differential calculus of functions of several variables is that, if you confine yourself to a set of small enough diameter, any reasonable function is approximately linear. If you really believe this, you are led to the conclusion that the area of of the part of the graph of $f(x, y)$ over a neighborhood of a point P_0 in the xy-plane can be approximated by considering the corresponding area in the tangent plane at the point of this surface above P_0. The sum of such areas for a partition of a region $\mathcal{D}$ in the xy-plane is a Riemann sum of

$$\iint_{\mathcal{D}} \sqrt{f_1(x, y)^2 + f_2(x, y)^2 + 1}\, dA. \qquad (A)$$

The quantity dA in this formula stands for $dx\,dy$ or $r\,dr\,d\theta$ in rectangular and polar coordinates, respectively.

A rigorous study of surface area is very difficult. Riemann integrals are defined in terms of a very general type of limit. The integrals exist under some fairly general assumptions, guaranteeing that the Riemann sums approximate the value of the integral found by calculus if the partition is fine enough. However, the Riemann sums just constructed involve approximating the surface by pieces that don't fit together to form an approximate surface. If we want to believe that this integral really does give surface area, it would be nice to connect it with the area of something that resembled the surface. A number of reasonable ideas for constructing such measurements turn out to be more general than Riemann sums and often fail to have limits. Although the values obtained from the integral (A) turn out to be correct whenever the integral makes sense, we cannot do a better job of relating them to geometric measurements of the surface.

There are only a few examples included in the exercises. The difficulty here is that most of the integrals obtained from (A) cannot be evaluated in terms of familiar functions. This difficulty was already present in connection with arc length. For example, the integral giving the perimeter of an ellipse usually cannot be expressed in terms of familiar functions. The use of (A) to set up an integral representing a surface area is one possible exercise. The warning, "Do not attempt to evaluate the integral", is given in such cases to signify that the result is not likely to be expressible in terms of familiar functions.

Surfaces of revolution. We show that this formula is consistent with the one used in section 10.3. For a surface of revolution given by a function, we have

$$z = f(r) \quad r^2 = x^2 + y^2.$$

Then $z_x = f'(r) r_x$ and $z_y = f'(r) r_y$ from the first equation, while $r_x = x/r$ and $r_y = y/r$ from the second. Thus, the area of the surface is given by integrating

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + \left(f'(r)\right)^2 \left(\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2\right)}$$
$$= \sqrt{1 + \left(f'(r)\right)^2}$$

with respect to area in xy-plane. Using polar coordinates, the inner integral is

$$\int \sqrt{1 + \left(f'(r)\right)^2}\, r\, dr$$

between appropriate values of r. This result is independent of θ, so the integration with respect to θ between 0 and 2π only multiplies this value by 2π.

Parametric surfaces. Section 16.6 extends the study of surface area to parametric surfaces.

Just as curves in space are best described by giving a vector function $\mathbf{r}(t)$ that may be thought of as describing how the curve is drawn, so surfaces should be given by a function $\mathbf{r}(u, v)$ expressing the space coordinates x, y, and z in terms of two parameters u and v that play the role of coordinates on the surface.

Surfaces of revolution are naturally parameterized by the adding an angular parameter θ to the parameter that draws the curve being rotated. If you are rotating about one of the coordinate axes, the distance along that axis is part of the parametric description of the

curve, and the other two coordinates are obtained by multiplying the distance to that axis (the other part of the parametric description) by $\cos\theta$ and $\sin\theta$.

Another family of surfaces that are easily obtained in *Maple* are the *tubeplots* the fatten up a space curve by identifying the points at a certain distance from the curve in the plane perpendicular to the curve at each point. One of the simplest examples if the **torus**. Start with the circle $r_0\langle\cos u, \sin u, 0\rangle$ and use the vectors $\langle\cos u, \sin u, 0\rangle$ and $\langle 0, 0, 1\rangle$ as perpendicular unit vectors in the normal plane, so one gets $(r_0 + r_1 \cos v)\langle\cos u, \sin u, 0\rangle + r_1 \sin v \langle 0, 0, 1\rangle$.

You can also get the surface traced out by the tangent lines to a space curve by letting $\mathbf{r}(u)$ be the curve and v the parameter that draws the tangent line at $r(u)$.

In the direct approach to area of parameterized surfaces, we partition the region in the uv-plane into pieces of small diameter, and approximate the corresponding part of the surface by parallelograms in the tangent plane whose sides are the given by vectors $\mathbf{r}_1(u, v)\Delta u$ and $\mathbf{r}_2(u, v)\Delta v$, the area of the parallelogram is given by the length of the cross product of

these vectors. As in our derivation of formula (A), this leads to

$$\iint |\mathbf{r}_1(u, v) \times \mathbf{r}_2(u, v)|\, du\, dv$$

over the region in the uv-plane parameterizing the part of the surface we are measuring. Comparing this to (A) in the case where x and y are functions of u and v and $z = f(x, y)$, we find

$$\mathbf{r}_1 = \left\langle x_u, y_u, f_x x_u + f_y y_u \right\rangle$$
$$\mathbf{r}_2 = \left\langle x_v, y_v, f_x x_v + f_y y_v \right\rangle$$

so that $\mathbf{r}_1(u, v) \times \mathbf{r}_2(u, v)$ is

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & f_x x_u + f_y y_u \\ x_v & y_v & f_x x_v + f_y y_v \end{vmatrix}$$

This leads to

$$du\, dv = \frac{dx\, dy}{|x_u y_v - x_v y_u|}.$$

The denominator is the third component of the cross product, and it is also the factor that will be found in section 15.9. In other words, this also corresponds to the formula that would be obtained by inventing the **implicit function** that gives $z = f(x, y)$ from the parameterization of the surface.

There is no special significance to the xy plane in these calculations. It is equally easy to use one of the other coordinate planes and the formulas will involve a different component of the vector perpendicular to the surface. Thus, if $\langle A, B, C\rangle$ is perpendicular to a surface at a point, then the element of surface area at that point dS satisfies

$$\frac{dS}{\sqrt{A^2 + B^2 + C^2}} = \left|\frac{dx\,dy}{C}\right| = \left|\frac{dx\,dz}{B}\right| = \left|\frac{dy\,dz}{A}\right|$$

Exercises 15.6

#3. Find the area of the portion of the plane $3x+2y+z = 6$ that lies in the first octant.

#9. Find the area of the potion of $z = xy$ that lies inside $x^2 + y^2 = 1$.