Area on a sphere. The ability to compute surface area allows us to measure familiar objects and refine our intuition about them. The formulas allow us to easily compute the area of the portion of the sphere between two parallels of latitude. Of course, the area of such a region is larger near the equator than near the pole, but a quantitative description will allow it to be compared to other measurements of the figure, and we will find a value for the area that is striking in its simplicity.

Following the conventions of *spherical and cylindrical coordinates*, a sphere of radius ρ will be formed by rotating a circle in an rz-plane (with the positive r-axis horizontal and pointing to the right and the positive z-axis pointing upward) of that radius about the z-axis. Using a convention different from geography, the circle will be parameterized using the angle ϕ measured clockwise from the positive z-axis. This gives

$$z = \rho \cos \phi \text{ and } r = \rho \sin \phi,$$

and the range $0 \le \phi \le \pi$ gives the semicircle in the right half plane. Rotating this semicircle all the

way around the z-axis (through an angle of 2π) gives the whole sphere. The region we want is given by rotating an arc between two fixed values of ϕ all the way around the z-axis. As with any problem involving figures of rotation, the parameterization of the surface will involve whatever parameter is used to describe the curve in the rz-plane that is to be rotated and a second parameter θ (longitude) giving the angle of rotation to reach the actual point on the surface. If the integration with respect to θ is done last, the symmetry of the surface assures us that it will be the integral of a constant from 0 to 2π, so that it will only multiply the value of the inner integral by 2π. This allowed many of these examples to be included in single variable calculus using a *hand-waving* argument to describe the role of rotation.

The parameterization of the surface (in rectangular coordinates) is

$$\langle \rho \sin \phi \cos \theta, \, \rho \sin \phi \sin \theta, \, \rho \cos \phi \rangle$$

and the partial derivatives with respect to ϕ and θ are

$$\langle \rho \cos \phi \cos \theta, \, \rho \cos \phi \sin \theta, \, -\rho \sin \phi \rangle,$$

$$\langle -\rho \sin \phi \sin \theta, \, \rho \sin \phi \cos \theta, \, 0 \rangle.$$

The cross product of these simplifies to

$$\left\langle \rho^2 \sin^2\phi \cos\theta,\, \rho^2 \sin^2\phi \sin\theta,\, \rho^2 \sin\phi \cos\phi \right\rangle.$$

which is a vector of length $\rho^2 \sin\phi$ in the radial direction. This length is the integrand in the surface area integral. Integrating with respect to ϕ gives the difference in values of $\rho^2 \cos\phi$ (since ρ is constant), and integrating with respect to θ multiplies by 2π as has already been noted. Thus the band between $\phi = \phi_0$ and $\phi = \phi_1$ has area

$$(2\pi\rho)(\rho\cos\phi_1 - \rho\cos\phi_0).$$

The first factor is the length of a great circle. The second factor is $z_1 - z_0$. In particular, if our sphere were surrounded by a cylinder of equal radius and our region projected to that cylinder from the common axis of the two figures, the spherical band and its projection on the cylinder would have equal area.

Integrals of vector fields over surfaces. Many physical applications involve vector fields interpreted as

flows and require the measurement of the **flux** through a surface. The contribution to the flow of a small piece of the surface should be the product of its area and the component of the flow perpendicular to it (since flow parallel to the surface does not cross it). There is an underlying assumption that the flow is a **vector** quantity and that it behaves in a **linear** fashion when vectors are added or multiplied by scalars. For anything with this behavior, it is essential that it be measured by a quantity that is sensitive only to normal components. These integrals are also **oriented**, in the sense that reversing the direction of the flow should give the negative of the previous measurement. This required that the surfaces have a clear **inside** and **outside**. In many of the exercises, the surface is the graph of a function $z = f(x, y)$ and an **upward** direction (i.e., a positive third component) can be used as a substitute for “outward”.

There are surfaces that do not have a global orientation. That it, you can walk around the surface carrying a continuously varying unit normal vector and get back to the same point with the normal having reversed its direction. The Möbius band is a common example: a rectangle is given a half-twist before gluing its ends together, so points that were on one side of the surface now find themselves next to points that were originally on the other side. We won't do anything with such surfaces except to acknowledge their existence.

In a later section, we will use surface integrals around closed surfaces to measure properties of the region inside the surface. For such results to make any sense, the surface must *have* an inside — i.e., it must have a global orientation. This is an extra concern in formulating theorems in this area, but it turns out not to cause any real trouble.

Computing surface integrals. When you want to do calculus, you determine what the expression you are computing looks like in the special case of the graph of a function. Since we are now concerned with surfaces, this means $z = f(x, y)$. We have already noted that a consistent orientation can be provided by the notion of **upward**. It remains only to convert the

flux integral

$$\iint \mathbf{F} \cdot \mathbf{n}\, dS$$

into something that we can compute.

Although the notation suggests that we should find $\mathbf{n}$ and dS separately and substitute our findings into the definition, these two terms should be considered as a single object. The reason for this is the formula from the last lecture, which in it oriented form reads

$$\frac{dS}{\sqrt{A^2 + B^2 + C^2}} = \frac{dx\, dy}{C} = \frac{dz\, dx}{B} = \frac{dy\, dz}{A}$$

where the quantities dx, dy, and dz are written in that cyclic order to find the order to be used for an adjacent pair. The homogeneous nature of this formula means that $\langle A, B, C\rangle$ can be taken to be any normal vector. If you multiply by $\langle A, B, C\rangle$, the first expression is exactly $\mathbf{n}\, dS$, and the others are the expressions to be used in computation. In one version, $\mathbf{F}$ is written as $\langle P, Q, R\rangle$ and the separate terms are written in the simplest form to get

$$\iint P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy. \qquad (*)$$

Another form which is suitable for graphs of functions used the vector with $C = 1$ as a normal to get

$$\iint \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dx\,dy.$$

The connection between these expressions is given by replacing dz by its expression in terms of dx and dy and treating $dx\,dx$ or $dy\,dy$ as zero, and $dy\,dx$ as $-dx\,dy$.

Exercises

Find $\iint \mathbf{F}\cdot\mathbf{n}\,dS$ over the given region (with upward orientation).

#19. $\mathbf{F} = \langle xy, yz, zx\rangle$ on $z = 4 - x^2 - y^2$ above $0 \le x \le 1$, $0 \le y \le 1$.

#21. $\mathbf{F} = \langle xye^y, -xze^y, z\rangle$ on the portion of $x + y + z = 1$ in the first octant.

An example using $(*)$ is

#27. $\mathbf{F} = \langle x, 2y, 3z\rangle$ on the surface of the cube with vertices $(\pm1, \pm1, \pm1)$ with outward orientation.

Another problem using unusual roles for the coordinate variables is

#25. $\mathbf{F} = \langle 0, y, -z\rangle$ with the outward orientation on the closed surface formed by the paraboloid $y = x^2 + z^2$ for $0 \le y \le 1$ and the disk $x^2 + z^2 \le 1$ in the plane $y = 1$.