

**Oriented integrals.** When the oriented surface integral is expressed in terms things like  $dx dy$  for computation, the computation is often guided by a picture in which the  $xy$ -plane was always viewed from above. Integrals of positive integrands that needed to be positive for physical reasons were positive because the limits of integration in each of the iterated integrals was written from left to right (or top to bottom), so that the result of the inner integral was a positive function whose integral would be positive. When extended to surfaces, this corresponded to an integral with an *upward* pointing normal. This agreed with the orientation of an *outward* pointing normal on a closed surface on the top of the surface and was opposite to that orientation on the bottom, so the total integral was the integral on the top *minus* the integral on the bottom, as those integrals were usually computed. In particular, the integral of  $z dx dy$  immediately becomes the integral of the difference of the values of  $z$  on the upper and lower boundaries of the region. Such an integral of *height* with respect to area can be interpreted as a *volume*. In the next part of the course, we will describe volume in a more fundamental way

as a *triple* integral, and the double integral of height will simply be the result of evaluating the innermost of the iterated integrals. The interpretation as a surface integral will lead to a proof of the *divergence theorem*, in exactly the same way as Green's theorem was based on interpreting the first part of the evaluation of a double integral as a line integral.

If the integral is to be calculated with respect to a different pair of coordinates, that coordinate plane is viewed from the positive side of the axis perpendicular to that plane, and a notion of *outward* needs to be converted to *positive* or *negative* by considering the component in this direction.

**Exterior algebra.** There is an algebraic approach to this in which  $dx dy$  is considered as a product of  $dx$  and  $dy$ . In this system, reversing the order of factors changes the sign of the product (just like the cross product of vectors). This can be useful when formulating the subject to replace all reliance on pictures with formal mathematics. One result of this is that it allows geometry in spaces of any number of dimensions, where the dimension is simply the number of

coordinates used to specify a point of the space. Anything in this theory that needs to be computed will be expressed as an iterated integral and computed one dimension at a time. When such integrals are formulated, there will be a convention to identify which endpoint gives the “lower” limit of integration. In order to avoid adding new orientation conventions to our techniques for working with multiple integrals, if we need to refer to this algebraic approach, the product of  $dx$  and  $dy$  in this system will be connected by an explicit  $\wedge$  symbol indicating this type of multiplication (much as *Maple* required an explicit  $*$  to indicate every ordinary multiplication).

**Applications of double integrals.** An easy way to claim that an integral arises as an application is to say that the integrand represents a **density**. This is mostly a matter of definition: if matter is spread unevenly over a plane region, it is reasonable to find an *average amount* over a piece of the region by cutting out the piece, weighing it, and dividing the weight by the area of the piece. If the piece is small enough, this will be close to a quantity that is *weight per unit area*

*at a point*. It is more accurate physically to treat the weight as an indirect measurement of *mass*. The same approach could be applied to other forms of physical *content*, like *electrical charge*, but I will use mass for most of the examples. It is a form of the fundamental theorem of calculus that the total content of the region is the integral of density. Thus, if there is some way in which the density is known, integration recovers to total content of a region.

Unfortunately, the density that is most likely to be known in advance is a *uniform* density. Then, mass is simply the product of area and density. The methods of single-variable calculus were often powerful enough to compute areas with a plausible argument that the integral was computing the intended area. Double integrals are much easier to justify, since they are based on a more refined type of Riemann sum, and they allow related computations to be formulated using a unified argument. In particular, the **moment** of a planar object with respect to a line is found by integrating the *signed distance* to that line with respect to mass. In single-variable calculus, this was

done when the line was one of the coordinate axes, but any such integral can be easily formulated and evaluated as a double integral. The textbook is a little timid about approaching this topic and continues to emphasize moments with respect to the coordinate axes. There are also **second moments**, for which the integrand is the *square* of the distance to a line (or a point). The physical term for a second moment is *moment of inertia*. One benefit of having simple expressions for these quantities is that one can prove properties of the moments by considering properties of the quantity that is integrated to obtain that moment.

**Theorems about moments.** Consider a region  $\mathcal{R}$  represent a physical object in the plane whose total mass is  $M$ .

**1.** If  $L_1$  is the line at signed distance  $d$  from line  $L_0$ , and the moment of  $\mathcal{R}$  about  $L_i$  is denoted  $\mu_i$ , then

$$\mu_0 = \mu_1 + Md$$

**1a.** Given  $L_0$ , one can find a parallel line  $L_1$  such that  $\mu_1 = 0$ .

**2.** If the moments of  $\mathcal{R}$  about two different lines through a point  $P$  are zero, then the moment of  $\mathcal{R}$  about any line through  $P$  is zero. The point  $P$  is called the **center of mass** of  $\mathcal{R}$ .

**3.** The second moment of  $\mathcal{R}$  about a line  $L$  is always nonnegative, so it can be denoted  $\sigma^2$ .

**3a.** If  $L_0$  passes through the center of mass of  $\mathcal{R}$  and  $L_1$  is parallel to  $L_0$  at distance  $d$ , and if the second moment of  $\mathcal{R}$  about  $L_i$  is  $\sigma_i^2$ , then

$$\sigma_1^2 = \sigma_0^2 + d^2.$$

**4.** The second moments about three different lines through the center of mass determine the second moments about all lines. There is an ellipse having the same second moments as  $\mathcal{R}$ .

A sufficiently general calculation of the center of mass of a figure of a particular shape gives a result which has a geometric interpretation. A few of these are worth remembering. For the immediate needs of exams in this course, knowing the answer will mainly be

useful as a check, since exam questions are likely to emphasize setting up the integral and showing some details of its calculation. However, once people accept that you have learned the calculus, there is little reason to do these calculations. Indeed, a poor choice of coordinates can give you an integral that is not easy to evaluate even though it has a simple value.

Example: Center of mass of a triangle.

Another advantage of reducing the calculation to an integral is that the integral may be evaluated by transforming to polar coordinates or using a special parameterization of the boundary to simplify the line integral arising from interpreting the integral as in the proof of Green's theorem.

Example: Center of mass of a circular sector.

**Probability.** Densities also appear in the study of probability. A single measurement typically has a **probability density function** on the line. The special features of such functions are that it is everywhere nonnegative and its integral over the whole line is 1. There was an introduction to this subject in Sec-

tion 8.5. With two measurements, a density function of ordered pairs of measurements can be defined. This is a nonnegative function on  $\mathbb{R}^2$  whose integral over the whole plane is 1. The first moments with respect to the axes are called **averages** and second moments with respect to a line through the center of mass are called **variance**.