Triple integrals. The definition of triple integrals involves Riemann sums based on partitioning intervals on the three coordinate axes into small subintervals. Such a definition is made to assure that certain physical an geometric quantities are given by integrals, and to prove the existence of the integral of functions that are continuous except of sets of lower dimension. This allows the theory to be based on integrals over rectangles with the extension to more general regions provided by multiplying he expected integrand by a function that is 1 on the domain of interest and 0 elsewhere. This is a wonderful theoretical tool, but it is never used in calculus.

Iterated integrals. Evaluating integrals in this course is done by using a geometric description of the region in which the section parallel to one of the axes is described. This works best when these intersections are always intervals. Then, the upper and lower endpoints are given as functions of the other two variables — where the intersection is nonempty. The set where there is a nonempty intersection is the projection of the three dimensional region into one of the coordinate planes. A similar description of such plane regions was given in the discussion of double integrals.

The volume of a region is **exactly** the integral of the constant 1 over the region. This allows maximal flexibility in choosing a strategy for evaluating the integral. Previous methods for finding volumes have involved **(1)** performing the first integration to find the height of the region in terms of its projection into one of the coordinate planes, or **(2)** reforming the first two integrations to find the area of a cross-section in terms of the coordinates along an axis. In these cases, the integration is done *by knowing the answer*, not by an explicit integration. However, such methods are **always** available. One of the things that makes calculus so useful is the organization that it provides for using results that have been computed previously. The first formulation of an integral is not always a good way to compute it, so we develop techniques for changing the order of integration, using different coordinate systems, or using vector calculus to replace given integrals by integrals along figures in a different dimension.

The subject is best described in term of examples.

Exercises.

In all cases, we are to integrate a function $f(x, y, z)$ over a region $\mathcal{E}$. When the region is given as the set of all (x, y, z) satisfying a list of inequalities, we just write the inequalities. If the integral is given explicitly as an iterated integral, we also write the inequalities.

#3. $f(x, y, z) = 6xz$, $\mathcal{E}$: $0 \le z \le 1$, $0 \le x \le z$, $0 \le y \le x + z$.

#5. $f(x, y, z) = ze^y$, $\mathcal{E}$: $0 \le y \le 3$, $0 \le z \le 1$, $0 \le x \le \sqrt{1 - z^2}$.

#7. $f(x, y, z) = 2x$, $\mathcal{E}$: $0 \le y \le 2$, $0 \le x \le \sqrt{4 - y^2}$, $0 \le z \le y$.

#9. $f(x, y, z) = 6xy$, $\mathcal{E}$ is under the plane $z = 1 + x + y$ and above region in the xy-plane bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$.

#11. $f(x, y, z) = xy$, $\mathcal{E}$ is solid tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$.

#13. $f(x, y, z) = z$, $\mathcal{E}$ is bounded by planes $x = 0$, $y = 0$, $z = 0$, $y + z = 1$, $x + z = 1$.

#15. $f(x, y, z) = x$, $\mathcal{E}$ is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

#19. $f(x, y, z) = 1$ (finding volume), $\mathcal{E}$ is bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 1$.

Finally, there is a problem the puts the ideas of integration together in a different way:

#47. Find the region $\mathcal{E}$ for which the integral over $\mathcal{E}$ of

$$1 - x^2 - 2y^2 - 3z^2$$

is a maximum.