Curl and Divergence. Stokes' Theorem expresses the integral of a vector field **F** around a closed curve as a surface integral of another vector field, called the **curl** of **F**. This vector field is constructed in the proof of the theorem. Once we have it, we invent the notation $\nabla \times \mathbf{F}$ in order to remember how to compute it. In this notation ∇ stands for the **vector operator**

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

and expressions containing ∇ are interpreted by first pretending that it is an ordinary vector, so that vector operations will introduce terms that abut a component from the first factor with one from the second factor. For numerical vectors, this is interpreted as multiplication, but for ∇ the interpretation is to let the component of ∇ operate on the component from the other factor.

The **gradient** also uses this interpretation of ∇ to construct a vector field from a scalar function. The **curl** takes one vector field to another using a construction modeled by the cross product.

There is something missing the we now fill in: $\nabla \cdot \mathbf{F}$ takes a vector field to a scalar function on $\mathbb{R}^3$. It is immediately clear how it is computed. All that must be added is that it has the name **divergence**.

Each of these acts as a derivative, and there is a version of the fundamental theorem that evaluates an integral of this derivative.

Special second derivatives. It has already been noted that

$$\nabla \times (\nabla f) = 0.$$

Formally, this is just the equality of mixed partials, but it is tied to Stokes' Theorem. If $\mathbf{F} = \nabla f$, the line integral of **F** along any curve is the difference of the values of f at the endpoints. For a closed curve, this is always zero. Stokes' Theorem then says that the surface integral of its curl is zero for every surface, so it is not surprising that the curl itself is zero.

Another consequence of the equality of mixed partials is that

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Stokes' Theorem says that the integral of $\nabla \times \mathbf{F}$ is zero over every closed oriented surface. The divergence theorem (to be described later) makes this quantitative by expressing the integral of a vector field over such a surface is equal to the integral of the divergence of that field over the region bounded by that surface. The proof will be similar to the proof of Green's theorem in the plane. This similarity suggests that Green's theorem could also be interpreted using a flux integral. This turns out not to be difficult: it is only necessary to rotate **T** into **N** and replace the vector field in the flow integral by the field the it rotates into under the same rotation.

Exercises 16.5. Find curl and divergence of the following vector fields.

#1. $\langle xy, yz, zx\rangle$.

#5. $\langle e^x \sin y, e^x \cos y, z\rangle$.

#7. $\left\langle \frac{x}{z}, \frac{y}{z}, \frac{1}{z}\right\rangle$.

For any of these whose curl is zero, express as a gradient.

Finding a vector field from its curl. The first four exercises of Section 16.8 have the form: use Stokes' Theorem to evaluate

$$\iint_{\mathbb{S}} \nabla \times \mathbf{F}\, d\mathbf{S}.$$

In this form, there isn't much to the exercise. This way of stating the exercise gives **F**, so its curl need never be computed since you must evaluate the line integral in order to feel that you have *used* Stokes' Theorem. There is still something to be done: you need to produce a parameterization of the positively oriented boundary from a description of the surface, but the statement of the exercise obscures its real content.

If a vector field **G** is of the form $\mathbf{G} = \nabla \times \mathbf{F}$, then $\nabla \cdot \mathbf{G} = 0$. For vector fields defined on all of $\mathbb{R}^3$, the converse is true. If we had a constructive proof of this result (as we do for the result that says that a vector field whose curl is zero is a gradient), then we would not need such a big hint to evaluate a surface integral by recognizing that it is given by an equivalent line integral. Since Stokes' Theorem is a consequence of Green's Theorem, and Green's Theorem is proved by showing that the evaluation of a double integral as an iterated integral leads to a line integral of a particular form, it is not unreasonable to look for a direct way to find the line integral appearing in Stokes' Theorem.

We now describe a solution of the equation

$$\nabla \times \mathbf{F} = \mathbf{G}$$

for **F** when **G** is given.

It is useful to simplify the problem as much as possible before beginning. We know that **F** is only defined up to a term of the form ∇f, and there is no no difficulty (in principle, although it does require integration) finding a function f for which $f_3(x, y, z)$ is any expression. In particular, any solution **F** could be replaced by $\mathbf{F} - \nabla f$, where $f_3(x, y, z)$ is equal to the third component of **F**. This means that we need only look for

$$\mathbf{F} = \langle X, Y, 0 \rangle.$$

Let

$$\mathbf{G} = \langle P, Q, R \rangle.$$

Looking at the components of the curl, we have

$$\begin{aligned} -Y_z &= P \\ X_z &= Q \\ Y_x - X_y &= R. \end{aligned} \qquad (*)$$

If the first of the equations in $(*)$ is solved for Y (up to a function of x and y), and this result put into the third equation, we are left with the task of finding X from its derivatives with respect to y and z. This was exactly what we did in recognizing conservative vector fields as gradients. The only requirement is that the equations $X_{zy} = X_{yz}$ must hold. These equations are

$$\begin{aligned} X_{zy} &= Q_y \\ X_{yz} &= Y_{xz} - R_z. \end{aligned}$$

Finally,

$$Y_{xz} = Y_{zx} = -P_x,$$

so

$$Q_y = -P_x - R_z.$$

This is exactly the given condition that $\nabla \cdot \mathbf{G} = 0$. In particular, any solution of the first equation of $(*)$ for Y leads to equations that can be solved for X.

An example. From exercise 1 in Section 16.5, we know that$\nabla \cdot \mathbf{G} = 0$ when $\mathbf{G} = \langle yz, xz, xy \rangle$. The first equation of $(*)$ becomes $Y_z = -yz$, so we take $Y = -yz^2/2$. This leaves

$$X_z = xz$$

$$X_y = -xy$$

(since $Y_x = 0$). Then $X = xz^2/2 + f(x,y)$ and $f_2(x,y) = -xy$. The solution can be completed because the expression that is supposed to be $f_2(x,y)$ really is independent of z. Integrating this, gives $f(x,y) = -xy^2/2$ as one solution. Thus,

$$\mathbf{F} = \frac{1}{2}\Big\langle xz^2 - xy^2, -yz^2, 0\Big\rangle.$$

If we subtract $\nabla(x^2z^2/4) = \big\langle xz^2/2, 0, x^2z/2\big\rangle$ from this, we have the more symmetric solution

$$\mathbf{F} = -\frac{1}{2}\Big\langle xy^2, yz^2, xz^2\Big\rangle.$$

The Divergence Theorem. The proof of the divergence is a three dimensional version of the proof of Green's theorem. It is proved first for regions for which all sections parallel to one of the coordinate axes are intervals (or empty). Evaluating a triple integral of an expression f over a solid body $\mathcal{B}$ as an iterated integral begins by using the fundamental theorem of one-variable calculus to find an expression R such that $\partial R/\partial z = f$. This is evaluated as integral over the projection in the xy plane of the difference of R for z at the top and bottom of $\mathcal{B}$. However, the operation of substituting the value of z on a surface and integrating R with respect to x and y is exactly the same as the surface integral of $R\mathbf{k}$ over the surface. Any part of the surface perpendicular to the xy plane gives a zero contribution to such a surface integral. The outward normal to the boundary $\mathcal{S}$ of $\mathcal{B}$, normalized to give an integral with respect to area in the xy plane has third component $+1$ on the top and third component -1 on the bottom. Since we have chosen a vector field whose first two components are zero, it is only the third component of the normal to the surface that contributes to the integral.

In the same way, on can integrate first with respect to x or y. This gives the three parts of the divergence theorem:

$$\iint_{\mathcal{S}} \langle P, 0, 0\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial P}{\partial x}\, dV$$
$$\iint_{\mathcal{S}} \langle 0, Q, 0\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial Q}{\partial y}\, dV$$
$$\iint_{\mathcal{S}} \langle 0, 0, R\rangle \cdot \mathbf{n}\, dS = \iiint_{\mathcal{B}} \frac{\partial R}{\partial z}\, dV$$

Adding these shows that the the integral of a vector field over $\mathcal{S}$ is equal to the integral of its divergence over $\mathcal{B}$.

The divergence theorem may be extended to more general regions that can by cut into pieces having the form used in the proof. Since the surface integral is oriented, its value on a cut will be counted with opposite signs when considered as part of the boundary of regions on the two sides of the cut. When one adds over the dissection of the region, the contribution of each cut simplifies to zero.

How is the divergence theorem used? Most exercises in the use of the divergence theorem, as is also true of the other theorems of vector calculus, use the triple integral to evaluate the surface integral. Since this is opposite to the direction of the proof, some explanation should be given. There are several reasons for this.

(1) It is certainly true, as in the proof of the divergence theorem, that the first step in the evaluation of a triple iterated integral over $\mathcal{B}$ can be interpreted as a surface integral over the boundary of $\mathcal{B}$. However, what the calculation actually gives is a double integral over the projection of $\mathcal{B}$ into one of the coordinate planes, which includes the use of a parameterization to begin the computation of the surface integral.

(2) The different components of the surface integral appear in the proof of the divergence theorem as the results of using different approaches to setting up an integral over $\mathcal{B}$ as in iterated integral. Normally, one only writes the integral in the one form that will be easiest to evaluate. When different terms of the triple integral are evaluated using different orders of inte-

gration, you are essentially choosing to assign those terms to different parts of the divergence.

(3) When the integrand in the surface integral is expressed in terms of polynomials, it becomes simpler when differentiated. Thus the integrand of the triple integral is simpler than that of the surface integral.

(4) Changes to cylindrical or spherical coordinates could be explained by using those coordinate systems to parameterize surfaces in surfaces integrals, but a direct description of triple integrals in these coordinate systems is often easier in those case where such coordinate systems are relevant.

(5) Certain triple integrals are known because they represent volumes or moments that are remembered from previous computations. It is not necessary to repeat these computations in order to get the answer, and the coordinate system of the current problem may make those computations tedious. Since the goal of these calculations is to get correct answers, some effort should be made to formulate problems in a way that simplifies computation.

Exercises 16.9 We will ignore specific instructions in the textbook and consider both sides of the divergence theorem identity for all examples. Typically, the vector field **F** to be integrated on the surface will be given, but the surface will be described only as the outwardly oriented boundary of a solid region.

#3. $\mathbf{F} = \langle 3x, xy, 2xz \rangle$. Region is cube in which each coordinate is between 0 and 1.

#5. $\mathbf{F} = \langle xy, yz, zx \rangle$. Region is solid cylinder defined by $x^2 + y^2 \leq 1$ and $0 \leq z \leq 1$.

#7. $\mathbf{F} = \left\langle 3x^2y^3, 9x^2yz^2, -4xy^2 \right\rangle$. Region is cube with vertices $(\pm 1, \pm 1, \pm 1)$.

#11. $\mathbf{F} = \left\langle 3xy^2, xe^z, z^3 \right\rangle$. Region is the cylinder bounded by $y^2 + z^2 = 1$, $x = -1$, $x = 2$.