Cylindrical and Spherical coordinates. If the rectangular coordinates are replaced by polar coordinates in the xy plane , the result is called **cylindrical coordinates**. Thus, the equations used are

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

Cylindrical coordinates give especially good descriptions of figures having rotational symmetry about the z axis. Such objects have equations that are independent of θ, and properties of the region can be found by drawing a picture in a plane with rectangular coordinates r and z. In many cases, it is possible to restrict attention to $r > 0$ since we allow θ to take all values from 0 to 2π, allowing all *rays* from the origin in the xy-plane to be described.

Replacing these rectangular coordinates by polar coordinates ρ and ϕ with the positive z axis as initial direction gives spherical coordinates. The equations

relating these systems are

$$z = \rho\cos\phi$$
$$r = \rho\sin\phi$$
$$\theta = \theta$$

Since we expect to consider only positive r, it is necessary only to consider $0 \le \phi \le \pi$ and $\rho \ge 0$.

Exercises 12.7

#51. Express $x + 2y + 3z = 6$ in other coordinate systems.

#55. Express $x^2 + y^2 = 2y$ in other coordinate systems.

#57. Describe $r^2 \le z^2 \le 2 - r^2$.

#59. Describe $-\pi/2 \le \theta \le \pi/2$, $0 \le \phi \le \pi/6$, $0 \le \rho \le \sec\phi$.

Integrals in these systems. The formulas needed to evaluate integrals are exactly those arising in polar coordinates: $dx\,dy = r\,dr\,d\theta$ becomes

$$dz\,dx\,dy = r\,dz\,dr\,d\theta$$

and $dz\,dr = \rho\,d\rho\,d\phi$ becomes

$$dz\,dr\,d\theta = \rho\,d\rho\,d\phi\,d\theta.$$

Combining these parts gives

$$dz\,dx\,dy = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.$$

This shows that it is possible to give formulas connecting rectangular and spherical coordinates, but it also shows that those formulas are direct consequences of the *much simpler* formula for relating rectangular and polar coordinates in a plane.

Another approach. The proof of the divergence theorem shows that the first step of evaluating an iterated integral gives a surface integral. For the integrals giving volumes or moments, the integrand is simple and so are all of its first integrals. If the surface is the graph of a function in cylindrical or spherical coordinates, then we can use the two independent variables

in that coordinate system to parameterize the surface. The first integrals in these curved coordinate systems may not agree exactly with the usual first integrals, but simple relations can be found. For example, the first integral of $r\,dr\,d\theta$ giving the area in polar coordinates inside $r = f(\theta)$ leads to the formula $\int f(\theta)^2/2\,d\theta$, which is the line integral of $r^2/2$ around the boundary of the region, that was used in a single-variable approach to this topic. To get such a simple expression starting from a line integral in rectangular coordinates, the integrand should be $\frac{1}{2}(x\,dy - y\,dx)$. Integration in polar coordinates automatically selects this more symmetric form of the area integral.

Exercises 15.8. In earlier exercises, the textbook tells you the system to use, which may be different from the system used to describe the region. Here, we will investigate all of the options. Also, in general, we will use only verbal descriptions since the notation for the integrals is only useful after you have identified the iterated integrals that you will be computing. Integration is always with respect to *volume*.

#7. Integrate $\sqrt{x^2 + y^2}$ over the region inside $x^2 +$

$y^2 = 16$ and between $z = -5$ and $z = 4$.

#11. Integrate x^2 over the region inside $x^2 + y^2 = 1$, above $z = 0$ and below $z^2 = 4x^2 + 4y^2$.

#17. Integrate $x^2+y^2+z^2$ over the ball $x^2+y^2+z^2 \leq 1$.

#21. Integrate $\sqrt{x^2 + y^2 + z^2}$ over the region that is bounded below by $\phi = \pi/6$ and bounded above by the $\rho = 2$.