FINAL EXAMINATION SOLUTIONS

1. Let $\mathbf{R}(t) = (e^t \cos t)\,\mathbf{i} + (e^t \sin t)\,\mathbf{j} + e^t\,\mathbf{k}$ give the position of a particle in space as a function of the time t, where $0 \le t \le \pi$.

(a) **Find the velocity** of the particle as a function of the time t. [10 pts.]

$$\frac{d\mathbf{R}(t)}{dt} = (e^t \cos t - e^t \sin t)\,\mathbf{i} + (e^t \sin t + e^t \cos t)\,\mathbf{j} + e^t\,\mathbf{k}\ .$$

(b) **Find the length of the arc** traversed by the moving particle for $0 \le t \le \pi$. [10 pts.]

$$\begin{aligned}
\left\|\frac{d\mathbf{R}(t)}{dt}\right\|^2 &= (e^{2t}\cos^2 t - 2e^{2t}\cos t \sin t + e^{2t}\sin^2 t) + (e^{2t}\sin^2 t + 2e^{2t}\cos t\sin t + e^{2t}\cos^2 t) + e^{2t} \\
&= 2e^{2t}(\sin^2 t + \cos^2 t) + e^{2t} = 2e^{2t} + e^{2t} = 3e^{2t} \\
\frac{ds}{dt} &= \left\|\frac{d\mathbf{R}(t)}{dt}\right\| = \sqrt{3e^{2t}} = \sqrt{3}\,e^t \\
s &= \int_0^\pi \frac{ds}{dt}\,dt = \int_0^\pi \sqrt{3}\,e^t\,dt = \sqrt{3}\cdot(e^\pi - 1)\ .
\end{aligned}$$

2. Find $\dfrac{\partial f}{\partial r}$ and $\dfrac{\partial f}{\partial s}$ as functions of r and s if

$$f(x,y) = x^2 + y^2$$

and the variables are related by $x = r - s$ and $y = r + s$. [15 pts.]

$$\begin{aligned}
&\frac{\partial x}{\partial r} = 1, \quad \frac{\partial x}{\partial s} = -1 \\
&\frac{\partial y}{\partial r} = 1, \quad \frac{\partial x}{\partial s} = 1 \\
&\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = 2x\cdot 1 + 2y \cdot 1 = 2(x+y) \quad (= 4r) \\
&\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2x\cdot(-1) + 2y\cdot 1 = 2(-x+y) \quad (= 4s)\ .
\end{aligned}$$

3. Find and classify the points at which local maxima and minima and saddles (if any) **occur** for the function

$$f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy\ .$$

[20 pts.]

The "critical points" at which maxima, minima or saddles occur are the solutions of the simultaneous equations

$$\begin{gathered}
\frac{\partial f}{\partial x} \equiv 12x - 6x^2 + 6y = 0 \quad \textit{and} \quad \frac{\partial f}{\partial y} \equiv 6y + 6x = 0 \\
2x - x^2 + y = 0 \quad \textit{and} \quad y = -x \\
x - x^2 = 0, \quad \textit{so} \quad x = 0 \ \textit{or}\ x = 1 \\
(x,y) = (0,0) \quad \textit{or} \quad (x,y) = (1,-1)\ .
\end{gathered}$$

The test for saddles vs. extrema is given by the sign of $f_{xx}f_{yy} - f_{xy}^2$. *The second partial derivatives of* f *are*

$$f_{xx}(x,y) = 12 - 12x, \quad f_{xy}(x,y) = f_{yx}(x,y) = 6 \quad \textit{and} \quad f_{yy}(x,y) = 6\ .$$

At $(0,0)$, $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 12\cdot 6 - 6^2 = 36 > 0$ *and the point is a (local) extremum; since* $f_{yy}(0,0) = 6 > 0$, *the point is a local minimum. At* $(1,-1)$, $f_{xx}(1,-1)f_{yy}(1,-1) - f_{xy}(1,-1)^2 = 0\cdot 6 - 6^2 = -36 < 0$ *and the point is a saddle.*

4. Let $f(x, y, z) = x^2 + y^2 - z^2 - 18$.

(a) **Explicitly compute** ∇f. [5 pts.]

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle 2x, 2y, -2z \rangle \ .$$

(b) **Find a normal** to the level surface $f(x, y, z) = 0$ at the point $(3, 5, -4)$, and give an equation of the plane tangent to that level surface at that point. [5 pts. + 5 pts.]

A normal to a level surface of a function at a point is given by the gradient of the function at the point, so a normal to this level surface is $\nabla f \Big]_{(3,5,-4)} = \langle 6, 10, 8 \rangle$, *or (if you wish)* $\langle 3, 5, 4 \rangle$. *The tangent plane of the level surface at that point must thus have the form* $3x + 5y + 4z = \text{const.}$, *and since the point of tangency* $(3, 5, -4)$ *must satisfy the equation of the tangent plane, the tangent plane is given by*

$$3x + 5y + 4z = 18 \ .$$

(c) **Compute the directional derivative** of $f(x, y, z)$ at the point $(1, 1, 1)$ in the direction in which the vector $-\mathbf{i} + \mathbf{j} - \mathbf{k} = \langle -1, 1, -1 \rangle$ points. [5 pts.]

The directional derivative of a function at a point in a given direction is given by the dot product of the gradient of the function, evaluated at that point, with a unit vector pointing in the given direction. In this setting we have $\nabla f \Big]_{(1,1,1)} = \langle 2, 2, -2 \rangle$, *the unit vector is* $\mathbf{u} = \frac{1}{\sqrt{3}} \langle -1, 1, -1 \rangle$, *and so the value of the directional derivative is* $\langle 2, 2, -2 \rangle \bullet \frac{1}{\sqrt{3}} \langle -1, 1, -1 \rangle = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$.

5.

(a) **Sketch the region in the first octant** of 3-dimensional space that is bounded above by the plane $z = x + y$ and on the sides and bottom by the coördinate planes and the elliptical cylinder

$$4x^2 + 9y^2 = 36 \ .$$

[10 pts.]

(b) **Express the volume of the region** sketched in (a) above as an iterated integral of the form

$$\int \int f(x, y) \, dy \, dx$$

with appropriate limits on the integrals. Do not evaluate the integral. [10 pts.]

$$\int_0^3 \int_0^{2\sqrt{9-x^2}/3} f(x, y) \, dy \, dx \ .$$

(c) **Express the volume of the region** sketched in (a) above as an iterated integral of the form

$$\int \int f(x, y) \, dx \, dy$$

with appropriate limits on the integrals. Do not evaluate the integral. [10 pts.]

$$\int_0^2 \int_0^{3\sqrt{4-y^2}/2} f(x, y) \, dx \, dy \ .$$

6. **Evaluate** $\displaystyle\int_0^2 \int_{y/2}^1 \frac{1}{(x^2+1)^3} \, dx \, dy$ by inverting the order of integration(1) and evaluating the new integral. [15 pts.]

$$\int_0^2 \int_{y/2}^1 \frac{1}{(x^2+1)^3} \, dx \, dy = \int_0^1 \int_0^{2x} \frac{1}{(x^2+1)^3} \, dy \, dx = \int_0^1 \frac{2x}{(x^2+1)^3} \, dx$$
$$= \frac{-1}{2} \frac{1}{(x^2+1)^2} \Bigg]_0^1 = \frac{-1}{2} \left[\frac{1}{4} - \frac{1}{1} \right] = \frac{3}{8} \ .$$

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(1) Credit will also be given for evaluating this integral by transforming it into polar coördinates, but the details of that method are more difficult to carry out.

7.

(a) **Make a first-octant sketch** of the region in 3-dimensional space bounded above by the sphere

$$x^2 + y^2 + z^2 = 1 \ ,$$

below by the xy-plane,(2) and on the sides by the cylinder

$$x^2 + y^2 = y \ . \qquad [5 \text{ pts.}]$$

(b) **Using cylindrical coördinates, express the volume of the entire region** described in (a) above (including that part lying over the 2nd quadrant of the xy-plane) as an iterated integral with respect to the variables r and θ. {Hint: Separately sketch the region of integration in the xy-plane; this will make the limits of integration, which must of course be expressed in polar coördinates, easier to find.} [10 pts.]

$$V = \int_0^{\pi} \int_0^{\sin\theta} \int_0^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta \ .$$

(c) **Evaluate the integral** of (b) above. [10 pts.]

Using symmetry, we see that

$$\begin{aligned} V &= 2\int_0^{\pi/2} \int_0^{\sin\theta} \int_0^{\sqrt{1-r^2}} dz\, r\, dr\, d\theta = 2\int_0^{\pi/2} \int_0^{\sin\theta} \sqrt{1-r^2}\, r\, dr\, d\theta \\ &= 2\int_0^{\pi/2} \left[\frac{-1}{2}\,\frac{2}{3}(1-r^2)^{3/2}\right]_{r=0}^{r=\sin\theta} d\theta = \frac{2}{3}\int_0^{\pi/2} \left[1-(1-\sin^2\theta)^{3/2}\right] d\theta \\ &= \frac{2}{3}\int_0^{\pi/2} [1-\cos^3\theta]\, d\theta = \frac{2}{3}\int_0^{\pi/2} [1-(1-\sin^2\theta)\cos\theta]\, d\theta = \frac{2}{3}\left[\theta - \sin\theta + \frac{\sin^3\theta}{3}\right]_0^{\pi/2} \\ &= \frac{2}{3}\left[\frac{\pi}{2} - 1 + \frac{1}{3}\right] = \frac{3\pi-4}{9} \ . \end{aligned}$$

If one does not use symmetry, one has to be careful to observe that $(1-\sin^2\theta)^{3/2} = [\cos^2\theta]^{3/2} = -\cos^3\theta$ *when* $\pi/2 < \theta \leq \pi$, *because* $\cos\theta$ *is negative for those values of* θ *and therefore* $\cos^3\theta$ *is negative also.*

8. Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 2$. {Hint: Even if you set up this problem in rectangular coördinates, you may want to shift to polar/cylindrical to evaluate the integral.} [15 pts.]

The element of surface area for the graph of the function $f(x, y) = x^2 + y^2$ *is given by*

$$dS = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\, dA = \sqrt{4x^2 + 4y^2 + 1}\, dA$$

where dA *is the element of area in the* xy*-plane; the part of the paraboloid whose area is sought lies over the region in the plane bounded by the solution of* $z = x^2 + y^2 = 2$, *which is a disc* R *of radius* $\sqrt{2}$ *centered at the origin. It is clear that cylindrical coördinates are well adapted to this problem. Thus the integral giving the area of this part of the paraboloid is*

$$\begin{aligned} \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dA &= \int_0^{2\pi} \int_0^{\sqrt{2}} \sqrt{4r^2 + 1}\, r\, dr\, d\theta \\ &= 2\pi \cdot \left[\frac{1}{8}\,\frac{2}{3}\,(4r^2 + 1)^{3/2}\right]_0^{\sqrt{2}} = \frac{\pi}{6}\,(27 - 1) = \frac{13\pi}{3} \ . \end{aligned}$$

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(2) The plane z=0.

9. Let $\mathbf{F}(x,y,z) = z^2\,\mathbf{i} + 2y\,\mathbf{j} + 2xz\,\mathbf{k}$. This vector field is defined and continuously differentiable everywhere in 3-dimensional space.

(a) **Determine whether there is a scalar function** $V(x,y,z)$ defined everywhere in 3-dimensional space, such that $\nabla V = \mathbf{F}$. If there is such a V, **find it**; if there is not, **explain why not**. [15 pts.]

Such functions V will exist, at least locally, if the curl of $\mathbf{F}$ is zero and can exist only if its curl is zero. Its curl is given by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 2y & 2xz \end{vmatrix} = (0-0)\mathbf{i} + (2z - 2z)\mathbf{j} + (0-0)\mathbf{k} = \mathbf{0}\,.$$

Since $V_x = z^2$ must hold, $V(x,y,z) = xz^2$ plus a function of y and z only. Since $V_y = 2y$ must hold, $V(x,y,z) = xz^2 + y^2$ plus a function of z only. Since $\dfrac{\partial(xz^2+2y)}{\partial z} = 2xz$, we must have $V(x,y,z) = xz^2 + y^2$ up to an additive constant.

(b) **Compute the line integral** $\int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the circular helix $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$, $0 \le t \le 2\pi$. Do not hesitate to use information developed in (a) above, if it will help. [10 pts.]

By the "Fundamental Theorem of Calculus for Line Integrals," we can compute the value of this line integral by evaluating the function $V(x,y,z)$ found in (a) above at the ending point $\mathbf{r}(2\pi) = (1,0,2\pi)$ and subtracting its value at the beginning point $\mathbf{r}(0) = (1,0,0)$. Thus

$$\int_C \mathbf{F} \bullet d\mathbf{r} = V(1,0,2\pi) - V(1,0,0) = [(1\cdot(2\pi)^2 + 0) - (0+0)] = 4\pi^2\,.$$

10. Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ that lies above the plane $z = 0$. The curve in which they intersect is the circle of radius 2 centered at the origin, which is easily parametrized using polar coördinates: $x(\theta) = 2\cos\theta$, $y(\theta) = 2\sin\theta$, $0 \le \theta \le 2\pi$. Let

$$\mathbf{F}(x,y,z) = (z-y)\,\mathbf{i} + (z+x)\,\mathbf{j} - (x+y)\,\mathbf{k}\,.$$

Compute $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\,dS$ by any means at your disposal. [15 pts.]

Stokes' theorem says that $\iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\,dS = \int_C \mathbf{F} \bullet d\mathbf{r}$, where C is the circle of radius 2 centered at the origin. This fact can be used directly, and the line integral around C can be computed by parametrizing the circle with the usual $x = 2\cos\theta$, $y = 2\sin\theta$, $z = 0$, yielding $dx = -2\sin\theta\,d\theta$, $dy = 2\cos\theta\,d\theta$, $dz = 0\,d\theta$ and so

$$\int_C \mathbf{F} \bullet d\mathbf{r} = 4\int_0^{2\pi} [(0 - \sin\theta)(-\sin\theta) + (0 + \cos\theta)(\cos\theta) + 0]\,d\theta = 4\int_0^{2\pi} [\sin^2\theta + \cos^2\theta]\,d\theta = 4\cdot 2\pi = 8\pi\,.$$

However, Stokes' theorem also implies that $\int_C \mathbf{F} \bullet d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \bullet \mathbf{n}\,dS$ where D is the disc in the xy-plane of radius 2 centered at the origin. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z-y & z+x & -x-y \end{vmatrix} = (-1-1)\mathbf{i} + (1-(-1))\mathbf{j} + (1-(-1))\mathbf{k} = -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$$

and the normal $\mathbf{n}$ to the xy-plane is $\mathbf{k}$, so the integrand $(\nabla \times \mathbf{F}) \bullet \mathbf{n} \equiv (-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \bullet \mathbf{k} \equiv 2$, identically constant. The integral over D therefore equals this constant times the area of D, or $2 \times 4\pi = 8\pi$.