MATH 251:10–12 FIRST HOUR EXAM SOLUTIONS

1 (a). The function

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}\ , \quad 0 \le t \le 2\pi$$

gives the position vector of a particle at time t. **Find the time or times in the given time interval** at which the velocity and acceleration vectors are orthogonal to one another. [6 pts.]

Differentiation gives

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = (1 - \cos t)\mathbf{i} + (\sin t)\mathbf{j} \qquad \mathbf{a}(t) = \frac{d^2\mathbf{r}(t)}{dt^2} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j}\ .$$

Testing vectors for orthogonality is done by the dot product (not the cross! which tests for parallelism): $\mathbf{x} \bullet \mathbf{y} = 0$ *holds if and only if the two vectors are orthogonal. So we compute*

$$\frac{d\mathbf{r}(t)}{dt} \bullet \frac{d^2\mathbf{r}(t)}{dt^2} = (1 - \cos t)\sin t + \sin t \cos t = \sin t\ .$$

The necessary and sufficient condition for orthogonality is thus that $\sin t = 0$*. For* $0 \le t \le 2\pi$*, this happens for* $t = 0$, $t = \pi$, *and* $t = 2\pi$.

1 (b). **Find the function** $\mathbf{r}(t)$ if its acceleration and its velocity and position at time $t = 0$ are respectively

$$\mathbf{a}(t) = 3t\mathbf{i} + 4\mathbf{j} + \mathbf{k},\ \mathbf{v}(0) = 4\mathbf{i},\ \mathbf{r}(0) = 5\mathbf{j}\ .$$ [6 pts.]

If $\mathbf{a}(t) = \dfrac{d^2\mathbf{r}(t)}{dt^2} = 3t\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, *then (taking an antiderivative of each term) we must have*

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \frac{3}{2}t^2\mathbf{i} + 4t\mathbf{j} + t\mathbf{k} + \mathbf{C}$$

where $\mathbf{C}$ *is a (vector) constant. Evaluation at* $t = 0$ *gives* $\mathbf{C} = \mathbf{v}(0) = 4\mathbf{i}$, *so*

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} = \left(\frac{3}{2}t^2 + 4\right)\mathbf{i} + 4t\mathbf{j} + t\mathbf{k}\ .$$

Taking antiderivatives again gives

$$\mathbf{r}(t) = \left(\frac{3}{6}t^3 + 4t\right)\mathbf{i} + 2t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k} + \mathbf{C}_1\ ,$$

and evaluation at $t = 0$ *gives* $\mathbf{C}_1 = \mathbf{r}(0) = 5\mathbf{j}$, *so finally*

$$\mathbf{r}(t) = \left(\frac{1}{2}t^3 + 4t\right)\mathbf{i} + (2t^2 + 5)\mathbf{j} + \frac{t^2}{2}\mathbf{k}\ .$$

2. In **(a)** and **(b)** below, let

$$z = f(x, y) = e^x \cos y\ .$$

(a): **Find** dz. [6 pts.]

Always $dz = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$, *so for this particular function* $f_x = e^x \cos y$, $f_y = -e^x \sin y$, *and thus*

$$dz = e^x \cos y\, dx - e^x \sin y\, dy\ .$$

(b): Suppose the measurements of x and y range in the intervals $-0.1 \leq x \leq 0.1$ and $-0.1 \leq y \leq 0.1$. **Use your result in (a) above to estimate the error** in the measured value $f(0,0) = 1$, using the estimates $e^x \leq 1.11$ and $|\cos y| \leq 1$. Your answer should be a (fairly small) number. [6 pts.]

The differential is used to make approximations ("tangent-plane approximations") to small changes in the function value corresponding to small changes in the arguments. In the present situation the small changes in x *and* y *could be at most* 0.1 *in absolute value. Consequently*

$$\begin{aligned} dz &= e^x \cos y\, dx - e^x \sin y\, dy = e^x \cos y \Big]_{(0,0)} dx - e^x \sin y \Big]_{(0,0)} dy = 1 \cdot dx + 0 \cdot dy \\ |dz| &= |dx| \leq 0.1 \ . \end{aligned}$$

To be super-cautious one might think that e^x *should be evaluated for the largest possible value of* x *with* $-0.1 \leq x \leq 0.1$ *and thus use* $e^{0.1} \approx 1.11$ *rather than* 1*, giving an error estimate of* 0.111*; either answer would have been accepted.*

3. Assuming that the equation

$$xy + z^3 x - 2yz = 0$$

defines z as a differentiable function of the two independent variables x and y near the point $(1,1,1)$, **find the value of** $\frac{\partial z}{\partial x}$ at the point $(x,y,z) = (1,1,1)$. [8 pts.]

One approach to this problem is simply to do what one did in the one-variable case: formally differentiate both sides of the equation with respect to x*, treating* z *as a function of* x *and (because it's partial differentiation)* y *as a constant. This would lead to the relations*

$$\begin{aligned} y + z^3 \cdot 1 + x \cdot 3z^2 \frac{\partial z}{\partial x} - 2y \frac{\partial z}{\partial x} &= 0 \\ y + z^3 &= (2y - 3xz^2) \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial x} &= \frac{y + z^3}{2y - 3xz^2} \\ \frac{\partial z}{\partial x}\Big]_{(1,1,1)} &= \frac{y + z^3}{2y - 3xz^2}\Big]_{(1,1,1)} = \frac{1+1}{2-3} = -2 \ . \end{aligned}$$

Another approach would plug in to the theorem: if $(x,y,z) = (1,1,1)$ *satisfies* $F(x,y,z) = 0$ *where* $F(x,y,z)$ *is a continuously differentiable function in an open set containing* $(1,1,1)$*, and if* $F_z(1,1,1) \neq 0$*, then the equation* $F(x,y,z) = 0$ *can be solved (uniquely) for a function* $z(x,y)$ *defined near* $(1,1)$*, taking the value* $z(1,1) = 1$*, and satisfying the equation* $F(x,y,z) = 0$ *identically, and its partial derivative with respect to* x *is given by*

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} \ .$$

In this case $F_x = y + z^3$ *and* $F_z = 3z^2 x - 2y$*, so*

$$\frac{\partial z}{\partial x} = -\frac{y + z^3}{3z^2 x - 2y} \ ,$$

the same expression we derived before. Evaluation at $(x,y,z) = (1,1,1)$ *leads to the same value for* $\frac{\partial z}{\partial x}$*.*

4. If $u = x^2 + e^{y^2}$, $x = \sin 2t$ and $y = \cos t^2$, **which of the following formulas give** $\dfrac{du}{dt}$? (There may be more than one, or none.) [6 pts.]

(a) $$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

(b) $$\frac{du}{dt} = 4x\cos 2t - 4ye^{y^2} t \sin t^2$$

(c) $$\frac{du}{dt} = 2x\cos 2t - 2ye^{y^2} \sin t^2$$

(d) None of the above.

(a) *must be correct: it's just the chain rule for plugging* $x(t)$ *and* $y(t)$ *into a function* $u(x, y)$. *If we compute the derivatives that get plugged into* (a) *in this context, we find that*

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{dx}{dt} = 2\cos 2t, \quad \frac{\partial u}{\partial y} = 2ye^{y^2} \quad \text{and} \quad \frac{dy}{dt} = -2t \sin t^2 \ .$$

Plugging these into (a) *gives* (b), *which is then also correct. But if* (b) *is right, then* (c) *is wrong; so the correct formulas are* (a) *and* (b) *only.*

5. Consider the plane curve

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j} \ .$$

(a): **Find the velocity and acceleration** of a point moving on this curve, **as functions of** t. Differentiate and simplify carefully; you will need the results below. ($\mathbf{v}(t)$ should be quite nice.) [8 pts.]

$$\begin{aligned} \mathbf{v}(t) &= \frac{d\mathbf{r}}{dt} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j} = t[(\cos t)\mathbf{i} + (\sin t)\mathbf{j}] \\ \mathbf{a}(t) &= \frac{d^2\mathbf{r}}{dt^2} = [(\cos t)\mathbf{i} + (\sin t)\mathbf{j}] + t[(-\sin t)\mathbf{i} + (\cos t)\mathbf{j}] \ . \end{aligned}$$

(b): **Find the length** of the portion of this curve corresponding to $0 \le t \le 2$. [8 pts.]

$$\left(\frac{ds}{dt}\right)^2 = \left\|\frac{d\mathbf{r}}{dt}\right\|^2 = t^2\|(\cos t)\mathbf{i} + (\sin t)\mathbf{j}\|^2 = t^2(\cos^2 t + \sin^2 t) = t^2, \text{ so}$$

$$\frac{ds}{dt} = \left\|\frac{d\mathbf{r}}{dt}\right\| = |t|, \quad s = \int_0^2 |t|\,dt = \int_0^2 t\,dt = \frac{t^2}{2}\bigg]_0^2 = 2 \ .$$

(c): **Find the unit tangent vector and curvature** of this curve, **as functions of** t. [6 pts.]

Because the velocity vector $\mathbf{v}(t) = t[(\cos t)\mathbf{i} + (\sin t)\mathbf{j}]$ *is just t-times the well-known unit vector* $(\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, *its length is simply* $|t|$, *and thus—for* $t > 0$—$\mathbf{T}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$. *Without reparametrizing the curve by arc length, one can find the curvature* $\kappa(t)$ *as a function of* t *by employing the formula* $\kappa(t) = \|\mathbf{r}'(t) \times \mathbf{r}''(t)\|/\|\mathbf{r}'(t)\|^3$. *Referring to the results of* **(a)** *above, we see that*

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t\cos t & t\sin t & 0 \\ \cos t - t\sin t & \sin t + t\cos t & 0 \end{vmatrix} = t^2\mathbf{k} \\ \|\mathbf{r}'(t) \times \mathbf{r}''(t)\| &= t^2 \\ \kappa(t) &= \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{t^2}{|t|^3} = \frac{1}{|t|} \ . \end{aligned}$$

Alternatively, since $\frac{ds}{dt} = t$, $s = \frac{t^2}{2}$, *and thus* $t = \sqrt{2s}$, *one can easily reparametrize this curve by arc length (almost impossible to do this explicitly for most curves) so that* $\mathbf{T}(s) = (\cos\sqrt{2s})\mathbf{i} + (\sin\sqrt{2s})\mathbf{j}$ *and thus*

$$\frac{d\mathbf{T}(s)}{ds} = \frac{1}{2\sqrt{2s}} \cdot 2 \cdot [(-\sin\sqrt{2s})\mathbf{i} + (\cos\sqrt{2s})\mathbf{j}]$$

$$\kappa = \left\| \frac{d\mathbf{T}(s)}{ds} \right\| = \frac{1}{\sqrt{2s}} = \frac{1}{t}\ ,$$

at least for $t > 0$.

6. Consider the function $f(x, y, z) = x^3 - xy^2 - z$.

(a). **Find the gradient** (field) ∇f. (A correct answer will contain functions of (x, y, z).) [8 pts.]

In general, $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$. *For this particular function,* $f_x = 3x^2 - y^2$, $f_y = -2xy$ *and* $f_z = -1$, *so*

$$\nabla f = \langle 3x^2 - y^2, -2xy, -1 \rangle = (3x^2 - y^2)\mathbf{i} - (2xy)\mathbf{j} - \mathbf{k}\ .$$

(b). **Find the directional derivative of** f at the point $(1, 1, 0)$ in the direction of $\langle 2, -3, 6 \rangle$. [8 pts.]

The directional derivative of f *at a point* P *in the direction of a unit vector* $\mathbf{u}$ *is given by the dot product* $\nabla f\big]_P \bullet \mathbf{u}$. *In this case the value of the gradient at* $(1, 1, 0)$ *is* $\langle 2, -2, -1 \rangle$, *a unit vector in the direction of* $\langle 2, -3, 6 \rangle$ *is* $\frac{1}{\sqrt{4+9+36}} \langle 2, -3, 6 \rangle = \left\langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \right\rangle$ *and the directional derivative is therefore*

$$\langle 2, -2, -1 \rangle \bullet \left\langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \right\rangle = \frac{4+6-6}{7} = \frac{4}{7}\ .$$

(c). **In the direction of what unit vector** does $f(x, y, z)$ increase most rapidly at $(1, 1, 0)$? [8 pts.]

The gradient of a function points in the direction of its most rapid increase. In this case, we know that value of the gradient at $(1, 1, 0)$ *is* $\langle 2, -2, -1 \rangle$, *and a unit vector in that direction is* $\mathbf{u} = \frac{1}{\sqrt{4+4+1}} \langle 2, -2, -1 \rangle = \left\langle \frac{2}{3}, \frac{-2}{3}, \frac{-1}{3} \right\rangle$.

7. **Find a function** $f(x, y, z)$ whose gradient is(1)

$$\nabla f = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$$

or **explain why no such function exists**. [8 pts.]

If this is the gradient of some function $f(x, y, z)$, *its components are given by* $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$, *and thus one ought to have* $f_x = 2y$, $f_y = 3x$ *and* $f_z = -z^2$. *The "cross-partials are equal" theorem would imply that then* $2 = f_{xy} = f_{yx} = 3$, *which is false. Since the partial derivative with respect to* y *of the* $\mathbf{i}$ *component does not equal the partial derivative with respect to* x *of the* $\mathbf{j}$ *component, this vector field cannot be the gradient of any function.*

8. **Find the cosine of the angle** between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$ at their line of intersection. [8 pts.]

Geometrically, the dot product of two vectors $\mathbf{a} \bullet \mathbf{b} = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta$, *where* θ *is the angle between the vectors. We can find the cosine of the angle between these two planes by realizing that this angle is the same as the angle between their normals. The normals are given by the vectors whose components are the coefficients:* $\langle 3, -6, -2 \rangle$ *and* $\langle 2, 1, -2 \rangle$ *respectively. Thus*

$$\cos\theta = \frac{\langle 3, -6, -2 \rangle \bullet \langle 2, 1, -2 \rangle}{\sqrt{9+36+4} \cdot \sqrt{4+1+4}} = \frac{6-6+4}{7 \cdot 3} = \frac{4}{21}\ .$$

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(1) Sigh. This question was supposed to be an 8-point no-brainer: this application of the "cross-partials are equal" theorem was pointedly reviewed in the last lecture before the hour exam.