MATH 251:10-11-12 SECOND HOUR EXAM SOLUTIONS

1. Find and classify the points at which local maxima and minima and saddles (if any) **occur** for the function

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4\ . \qquad [15 \text{ pts.}]$$

The first and second partial derivatives are given by

$$\frac{\partial f}{\partial x} = y - 2x - 2, \qquad \frac{\partial f}{\partial y} = x - 2y - 2$$
$$f_{xx} = -2, \quad f_{xy} = f_{yx} = 1, \quad f_{yy} = -2\ .$$

Setting $f_x = 0$ and $f_y = 0$ gives the simultaneous linear equations $-2x + y = 2$ and $x - 2y = 2$ with solution $(x,y) = (-2,-2)$. At this point $f_{xx}f_{yy} - fxy^2 = (-2)(-2) - 1^2 = 3 > 0$ so this critical point is not a saddle, and since both $f_{xx} < 0$ and $f_{yy} < 0$, the point is a (local, but actually absolute) maximum.

2. Find the absolute maximum and minimum values of the function $f(x,y) = xy$ on the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$. This problem may be done either by the method of Lagrange multipliers or by the method of parametrizing the curve (and thus reducing the problem to a one-variable extremum problem). [10 pts.]

There are at least three ways to do this problem. (1) One can use the method of Lagrange multipliers. We have $\nabla f = \langle y, x\rangle$ and $\nabla g = \left\langle \dfrac{2x}{9}, \dfrac{2y}{4}\right\rangle$, so at any local constrained max or min there must exist λ for which $\nabla f = \lambda \nabla g$, giving the two equations $y = \lambda \cdot 2x/9$ and $x = \lambda \cdot 2y/4$. The absolute max value must be positive and the absolute min value must be negative, so neither $x = 0$ nor $y = 0$ can hold at an extremum and thus it is legitimate to divide one of these equations by the other, giving $\dfrac{y}{x} = \dfrac{2\lambda x}{9}\dfrac{4}{2\lambda y} = \dfrac{4x}{9y}$, or $y^2 = \dfrac{4}{9}x^2$. Plugging this into the constraint equation gives $\dfrac{x^2}{9} + \dfrac{1}{4}\dfrac{4x^2}{9} = 1$ or $x^2 = \dfrac{9}{2}$. This gives $x = \dfrac{\pm 3\sqrt{2}}{2}$ and $y = \pm\sqrt{2}$. The maximum value of $f(x,y) = xy$ is thus 3 and the minimum value -3. (2) One can parametrize the ellipse using $x = 3\cos t$ and $y = 2\sin t$. Then $f(x(t), y(t)) = 6\sin t\cos t = 3\sin 2t$, the derivative of which is $6\cos 2t$. The zeros of $\cos\theta$ occur at $\theta = \pm\pi/2$ and $\theta = \pm 3\pi/2$ and thus the zeros of the derivative occur at $t = \pm\pi/4$ and $t = \pm 3\pi/4$. At the first of those points one has $x = 3\cdot(\sqrt{2}/2)$ and $y = \pm\sqrt{2}/2$, exhibiting the maximum value 3 and minimum value -3, and the same values occur at the other two points (up to a change in sign). (3) One can solve the constraint equation for one of the variables in terms of the other, do a single-variable max-min problem and check the endpoints. For example, the constraint gives $y^2 = 4[1 - (x^2/9)]$ or $y = 2\sqrt{1 - \dfrac{x^2}{9}}$, so that in terms of x only one has

$$\begin{aligned} f(x) &= x \cdot 2\sqrt{1 - \frac{x^2}{9}} \\ f'(x) &= 2\left[\left(1 - \frac{x^2}{9}\right)^{1/2} + x \cdot \frac{1}{2}\left(1 - \frac{x^2}{9}\right)^{-1/2} \cdot \left(\frac{-2x}{9}\right)\right] \\ &= 2\left(1 - \frac{x^2}{9}\right)^{-1/2}\left[1 - \frac{x^2}{9} - \frac{x^2}{9}\right]\ . \end{aligned}$$

The zeros of the derivative occur where $x^2 = \dfrac{9}{2}$. which leads to the same results as we saw before. The end-point check gives the value zero, obviously not an extremum.

3. **Sketch the region** in the xy-plane over which the integration

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx$$

takes place, and **write an iterated integral** of the form $\int_a^b \int_{g_1(y)}^{g_2(y)} (4x+2)\,dx\,dy$ that represents the same double integral. **Evaluate one (ONLY)** of the integrals. [15 pts.]

The region of integration is bounded by the line $y=2x$ on the left or top and the parabola $y=x^2$ on the right or bottom. This boundary, written in terms of functions $x=g(y)$, is given by the line $x=y/2$ and the parabola $x=\sqrt{y}$ respectively. Thus the corresponding iterated integral in reverse order is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy\ .$$

Depending on which order of integration one chooses, one has

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx = \int_0^2 (4x+2)(2x-x^2)\,dx = \Big[2x^2+2x^3-x^4\Big]_0^2 = 8$$

or

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x+2)\,dx\,dy = \int_0^4 \Big[2x^2+2x\Big]_{y/2}^{\sqrt{y}}\,dy = \int_0^4 \left[y+2\sqrt{y}-\frac{y^2}{2}\right] dy$$
$$= \left[\frac{y^2}{2}+\frac{4}{3}y^{3/2}-\frac{y^3}{6}\right]_0^4 = 8+\frac{32}{3}-\frac{32}{3} = 8\ .$$

4. Consider the integral $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$. It is impossible to evaluate this integral in rectangular coördinates. **Sketch** the region of integration, **write an integral in polar coördinates** that represents the same double integral, and **evaluate the polar-coördinate integral**. {Note: the region is bounded by the x-axis and the circle of radius 1 centered at the origin, and lies above the x-axis.} [15 pts.]

The description of the shape of the region tells us the limits of integration in polar coördinates, and so

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^\pi \int_0^1 e^{r^2}\,r\,dr\,d\theta = \pi\cdot\left[\frac{e^{r^2}}{2}\right]_{r=0}^{r=1} = \frac{\pi\cdot(e-1)}{2}\ .$$

5. **Set up, and then evaluate**, a triple integral in rectangular or cylindrical coördinates (your choice) whose value gives the volume of the "wedge"-shaped solid which the plane $z=0$ (bottom) and the plane $z=y$ (top) cut from the cylinder bounded by $x^2+y^2=1$. **Make a sketch** of the part of the solid lying in the first octant (but be sure your integral includes all of the solid). [15 pts.]

The integral takes one of the forms

$$V = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^y dz\,dy\,dx \quad \text{or}$$
$$V = \int_0^\pi \int_0^1 \int_0^{r\sin\theta} dz\,r\,dr\,d\theta$$

according to the setup you chose. This gives

$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} \int_0^y dz\,dy\,dx = \int_{-1}^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_{-1}^1 \frac{1-x^2}{2}\,dx = \frac{x}{2}-\frac{x^3}{6}\Big]_{-1}^1 = 2/3 \quad \text{or}$$
$$\int_0^\pi \int_0^1 \int_0^{r\sin\theta} dz\,r\,dr\,d\theta = \int_0^\pi \int_0^1 \sin\theta\,r^2\,dr\,d\theta = \int_0^\pi \frac{1}{3}\sin\theta\,d\theta = \frac{-\cos\theta}{3}\Big]_0^\pi = \frac{2}{3}\ .$$

6. **Sketch the region** described in spherical coördinates as follows: the region inside the sphere $\rho = a$ (here a is a positive constant, the radius of a certain sphere) and between the cones $\phi = \dfrac{\pi}{3}$ and $\phi = \dfrac{2\pi}{3}$. Then **find the volume of the region** by triple integration in spherical coördinates. The element of volume in spherical coördinates is $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. [15 pts.]

The region is a sphere of radius a with a (double) right circular cone cut out of it, the vertex of the cone being at the center of the sphere and the sides of the cone making an angle of $\frac{\pi}{3}$ with its axis. Thus

$$V = \int_0^{2\pi} \int_{\pi/3}^{2\pi/3} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{2\pi a^3}{3} \int_{\pi/3}^{2\pi/3} \sin\phi \, d\phi = \frac{2\pi a^3}{3} \cdot [-\cos(2\pi/3) + \cos(\pi/3)] = \frac{2\pi a^3}{3} \, ,$$

which is (perhaps amusingly) half the volume of the whole sphere.

7. **Evaluate the line integral**

$$\int_C (1 - y^2) \, dx - 2xy \, dy$$

where C is the parametrized curve $x(t) = -\cos t$, $y(t) = \sin t$ and t runs from 0 to π. [15 pts.]

Since $\dfrac{\partial(1-y^2)}{\partial y} = -2y = \dfrac{\partial(-2xy)}{\partial x}$, this integral passes the "cross-partials-are-equal" test, and indeed, for $u(x,y) = x - xy^2$ we have $\dfrac{\partial u}{\partial x} = 1 - y^2$ and $\dfrac{\partial u}{\partial y} = -2xy$. By the fundamental theorem of calculus for line integrals,

$$\int_C (1 - y^2) \, dx - 2xy \, dy = u(-\cos\pi, \sin\pi) - u(-\cos 0, \sin 0) = u(1,0) - u(-1,0) = (1-0) - (-1-0) = 2.$$

However, it is possible to compute the line integral directly: one has $dx = \sin t \, dt$ and $dy = \cos t \, dt$ and thus

$$\int_C (1 - y^2) \, dx - 2xy \, dy = \int_0^\pi [(1 - \sin^2 t) \sin t + 2 \sin t \cos^2 t] \, dt = \int_0^\pi [(\cos^2 t) \sin t + 2 \sin t \cos^2 t] \, dt$$
$$= \int_0^\pi 3 \cos^2 t \sin t \, dt = -\cos^3 t \Big]_0^\pi = -(-1) - (-1) = 2 \, .$$

It might be possible to use Green's theorem in the evaluation in some way, but would probably not be worth the effort.