# Math 251:10–12 — Spring 1999
# MW6 CHM-106
# Prof. Bumby

**This section on the Web**. Copies of all handouts are being made available within a week of distribution in class on a page linked to my personal home page. Point your browser at

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**Overview of part 2 of the course: February 10–March 01**. This chapter deals with the differential calculus of functions of several variables. One of the main ingredients is the *partial derivative*, but this should already be familiar to you. When you ask *Maple* to differentiate an expression, you are required to name the variable with respect to which you want to perform the differentiation because other variables may be present in the expression (perhaps by accident); such additional variables will be treated as constants. This is so natural that it may have escaped your notice. A more complicated situation where functions of several variables have already appeared is *implicit differentiation*. A typical explanation of this subject begins: **pretend** that you know a function expressing $y$ in terms of $x$ that satisfies the relation $f(x, y) = 0$. Then **differentiate**. Since $f(x, y)$ is supposed to be **identically** zero, its derivative is also zero. On the other hand the expression for $f(x, y)$ can be differentiated with respect to $x$, and the **chain rule** will introduce a factor of $dy/dx$ whenever an expression containing $y$ is differentiated. Indeed, the resulting expression turns out to be **linear** in $dy/dx$, so it is easy to solve for $dy/dx$ in terms of $x$ and $y$. One of our results will be an expression for this solution in terms of the partial derivatives of $f$ with respect to $x$ and $y$. The proof of this formula uses a **chain rule** for functions of two variables.

To put all of this on a firm foundation, some sort of **total** derivative of functions of several derivatives is constructed. The theoretical side of things is a little worse than in the case of functions of one variable, because there are many ways in which such functions can appear reasonable but behave badly. In particular, consider the function $x/y$ or $x^2/(x^2 + y^2)$. Both of these fail to be defined at the origin (indeed the former cannot be defined at any point where $y = 0$ since evaluation of the function involves division by $y$, but the latter is defined everywhere except at the origin). Our definition of limits of functions of several variables will say that this function has no limit as the origin; but restricting the function to a straight line through the origin yields a constant function. However, different lines give different constants. Thus, approaching the origin along a single line, or even a smooth curve, suggests that there is a limit, but different lines suggest different limits. The best way to resolve such conflicts is to allow a function $f(x, y)$ to fail to have a limit at a point $P$ unless all points $(x, y)$ close to $P$ lead to values of $f(x, y)$ that are close together. The resulting definition of continuity is easy to verify for many functions, and the existence of continuous partial derivatives is frequently sufficient to guarantee that the calculus works as expected. This means that getting reasonable answers from the formulas of calculus automatically proves that those answers are correct — just as in single variable calculus.

An important application of differential calculus is the location of maxima and minima of functions. The theorem that a continuous function on a closed and bounded set has extreme values and those values are attained is true in this generality. For an interior point of the region, it is easy to show that, if our total derivative is not zero, the point cannot be the location of a maximum or minimum. However, the theorem applies only to closed regions, so it is necessary to study the boundary as well. In one-variable calculus, we usually deal with problems in which the domain of the function is a closed interval, whose boundary consists of two points. Adding these points to the list of candidates for the location of extreme values is an efficient way to locate the extreme values. In single variable calculus, one can be a little sloppy about this, since the interesting problems usually have rather dull behavior on the boundary. However, the opposite tends to be true for functions of several variables. Not only does the boundary contain infinitely many points, but many important functions have such dull behavior on the interior of the region that extrema are frequently located only on boundaries. The method for locating such extrema is called the method of *Lagrange multipliers*, described in the last section of chapter 12.

The exam for this part of the course is scheduled for **March 01**.

**Maple Labs**. Maple Lab #1 is being distributed with this handout on the day of the first exam — February 08. The normal time allowed for these labs is two weeks, so you should be finished by February 22 (although labs will be accepted at the February 24 lecture and February 25 recitation also). It is expected that Maple Lab #2 will be distributed on February 22 to allow it to be finished before Spring Break.

**Details**. Here are the topics to be covered in lectures on Chapter 12 leading to the exam on March 01.

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