Math 251:10–12 — Spring 1999
MW6 CHM-106
Prof. Bumby
Lab 2 QUADRIC SURFACES

This description of the lab is only for the sections identified in the header. Other sections are expected to use a different version.

Please turn in only the printout of your Maple worksheets. Use the **text** feature of Maple to add a header containing your name, identify the separate parts of the lab, labels for the plots in your worksheet and explicit answers to all questions asked (do not write this material in by hand). Remove from the worksheet any extraneous material such as the output of commands like `with(plots);` (or, better, use a colon rather than a semicolon at the end of such commands to suppress output) and any errors you have made. To be sure that the input and output shown agree, you should use a `restart` command and then select **Execute Worksheet** from the Edit Menu.

You can save yourself typing, and have reliable forms of instructions given here, by first copying prepared Maple worksheet files to your home directory. This is done by going to the course page (if you haven't created a bookmark for it, go to **http://www.math.rutgers.edu/~bumby** and follow the link to this course), skipping to the bottom of the page, and downloading the "seed file" for lab2. If you don't see this link, you have probably cached a previous version of the page, and you should ask your browser to *refresh* the file; if your browser shows you the file, just select *Save As*. . . from the File Menu to get a copy. Use only these files, not those designed for other sections that you may find elsewhere.

The large number of graphs have caused *Maple* to crash when they were combined into a single worksheet. Use separate worksheets as indicated below. You should also save your work often. This is good practice in general, but especially important here, in case we have not completely identified the cause of *Maple*'s distress.

A quadric surface is the graph of a second-degree equation in three variables x, y, and z. The most general form of such an equation is:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, D, E, F, G, H, I$, and J are constants. In this lab, we use Maple to help visualize some types of quadric surfaces that can arise from equations of the above form.

There are three Maple commands that we will use to plot surfaces in three dimensions. These are `plot3d`, `implicitplot3d`, and `display`. The first is always available, but before the second or third of these can be used, one must type the command `with(plots):`. The command `plot3d` may be used to graph a surface when one of the variables can be solved for explicitly in terms of the other two, e.g., $z = f(x, y)$, while the command `implicitplot3d` is needed to graph any surface of the form $g(x, y, z) = 0$. Any surface of the first form is also of the second form, taking $g(x, y, z) = z - f(x, y)$, so the `implicitplot3d` command is more general. The command `display` is used to display two previously defined plots at the same time on the same set of axes. Be sure to end a command that creates a named plot for later display with a colon: you really don't want the output it generates. (It won't hurt to see it *once*, but it should not be part of the report you submit.)

The surfaces $z = x^2 + y^2$ and $z = 4$ may be plotted at the same time by using the syntax

```
plot3d({x^2 + y^2, 4}, x = 0..1, y = 0..1);.
```

However, this limits your options. We are interested in identifying the intersection of two surfaces, so we shall adopt an alternative approach using the `display` command which allows us to choose a different color for each plot. This also allows plots generated by `plot3d` and `implicitplot3d` to be combined in the same display. The `setoptions3d` instruction assures that some properties are made part of every plot without special request. Here, we request the boxed form of coordinate axes, equal scales on all axes, and solid colors, since these options are most useful for the questions in the lab.

0. Some *Maple* commands are introduced. You are mostly only required to observe, but part (c) asks for an interpretation of the graphs that were drawn.

(a) Execute the following sequence of commands.

```
with(plots):
setoptions3d(axes=BOXED,scaling=CONSTRAINED,style=PATCH);
p1:=plot3d(1, y=0..1,z=0..2, color=RED):
p2:=plot3d(2, x=0..1,y=0..1, color=BLUE):
display({p1,p2});
```

The colors won't appear when printed in black and white, so add some text to describe the color of different parts of the graph. Then,

(b) execute these commands.

```
p3:=implicitplot3d(x=1, x=0..1, y=0..1,z=0..2, color=RED):
p4:=implicitplot3d(z=2, x=0..1,y=0..1,z=0..2, color=BLUE):
display({p3,p4});
```

Again, describe the graph. Also,

(c) insert some **text** analyzing why such similar commands give such different graphs.

Note. One problem with this use of the `display` command is that you do not see any of the plots until the `display` command is used. You can get around this by first typing

```
plot3d(1, y=0..1,z=0..2, color=RED);
```

to view the plot. If it is satisfactory, edit the line by inserting `p1:=` at the beginning and changing the semicolon at the end to a colon, and then execute it again. This names the plot and suppresses all output.

1. An ellipsoid is a quadric surface of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{1}$$

where a, b, and c are positive constants. In this problem, we consider the ellipsoid with $a = 2, b = 3, c = 5$. By giving *Maple* these values, it is possible to use general descriptions that will have specific interpretations when the worksheet is executed. You can use the same worksheet for another example by changing the definitions of a, b, and c and re-execution the worksheet.

(a) One approach to plotting the surface given by (1) is to solve for z, obtaining the two surfaces

$$z = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}, \qquad z = -c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

which correspond to the top and bottom half of the ellipsoid. Each of these surfaces is defined explicitly in the form $z = f(x, y)$ and thus can be graphed using the `plot3d` command. The `plot3d` command requires you to specify a domain over which you wish to plot the graph. For example, `top:=c*sqrt(1-x^2/a^2-y^2/b^2);` is an expression for the top half of the ellipsoid, and `plot3d(top,x=-a..a,y=-b..b);` is a request to plot its graph over a rectangular region containing all points for which the expression `top` can be defined.

It is also possible to construct plots over domains that are not rectangles. This is accomplished by giving bounds on the second variable that depend on the first variable, although it is still required that the bounds on the first variable be constant (at the time that the `plot3d` command is executed. This description of regions in the plane will be useful later in the course, so you should practice it with *Maple*. The exact domain for which the expression `top` yields real values is

$$-a \le x \le a; \qquad -b\sqrt{1-(x^2/a^2)} \le y \le b\sqrt{1-(x^2/a^2)}.$$

The "seed file" contains instructions for obtaining three variations on this plot. If the plots are the same size (and not too small, please), they should be identical.

(b) The top and bottom of the ellipsoid can be plotted together in the same `plot3d` command. The whole ellipsoid can also be plotted using the `implicitplot3d` command. The "seed file" contains instructions for obtaining plots in both styles. Two descriptions of the domain are used with `plot3d`, but the `implicitplot3d` command requires constant bounds. Execute these commands to obtain three plots. Note that the name $E1$ is introduced for one of the plots to allow it to be used in later `display` commands. Adjust the size of the graphs and add labels. Finally, describe any differences between these plots.

(c) Use the instructions in the "seed file" to find the intersection of the ellipsoid $E1$ with the plane $x = 1$. Rotate the plot (and adjust the size) until it clearly shows the curve of intersection. Then, construct a variant to find the intersection of $E1$ with the plane $y = 2$, and a separate plot to show the intersection of $E1$ with the plane $z = 3$.

(d) The intersection of $E1$ with the plane $y = 2$ can easily be found without *Maple*. Do the algebra to find an equation for the intersection, and use this equation to identify the curve. Use the **text** feature of Maple to insert this information. Does this agree with the plots obtained by *Maple*?

Each of the remaining problems should be done on separate worksheets with distinct names. When you switch worksheets within the same *Maple* session, you should execute a `restart` command. Breaking the work into several worksheets also allows independent parts of the project to be done at different times. This level of independence is the best way to avoid difficulties caused by limited machine resources.

2. The surface

$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \tag{2}$$

is called an elliptic paraboloid.

Again, set $a = 2$, $b = 3$, $c = 5$, and create plots over the region $-2 \le x \le 2$, $-3 \le y \le 3$.

(a) Solve (2) for z in terms of x and y and use the `plot3d` command. What are the values of z shown in this plot?

(b) Now put the original form of (2) into the `implicitplot3d` command with $-2 \le x \le 2$, $-3 \le y \le 3$ and the range of z found in (a). Is there a difference between these two plots? If so, which do you prefer?

(c) Obtain three plots which clearly show the intersection of this paraboloid with the planes $x = 1$, $y = 2$, $z = 3$, respectively.

(d) Determine without the use of *Maple* the equation of the curve of intersection for each of the planes in (c) and use the **text** feature of Maple to enter it into your Maple worksheet, making sure to label your equation (e.g., Equation of curve of intersection with (x=1)). Also state explicitly what type of curve this is.

3. The surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \tag{3}$$

is known as an (elliptic) hyperboloid of one sheet. Again, set $a = 2$, $b = 3$, $c = 5$. This time, we see from (3) that the surface can only be defined for (x, y) *outside* the ellipse $x^2/a^2 + y^2/b^2 = 1$, so a larger domain is needed in order to get a good view of the surface, so we choose

$$-2a \leq x \leq 2a, \quad -2b \leq y \leq 2b, \quad -2c \leq z \leq 2c. \tag{3a}$$

(a) Use `implicitplot3d` to obtain a graph of the surface with equation (3) with the bounds given by $(3a)$. As in earlier examples, add a `color` option to the `implicitplot3d` command.

(b) Now use `plot3d` to plot the plane

$$z = \frac{c}{b}y + 5x - 5a \tag{3b}$$

for $-2a \leq x \leq 2a$ and $-2b \leq y \leq 2b$ using a different color. Insert **text** giving the range of this function on this domain. Then, edit this command, adding the *option* `view=-2*c..2c` to the arguments of the `plot3d` command (see *help* for `plot3d` and `plot3d,option` for details), assigning this plot a name, and replacing the final semicolon with a colon. This will remove a plot from your worksheet whose only interesting feature has been recorded as **text**.

Now `display` together the graphs you saved of (3) and (3b), rotating the plot to get a view of the intersection.

(c) In the same way, add the plane

$$z = \frac{c}{b}y - x + a \tag{3c}$$

to your plot, using a third color. The algebra needed to show that the line of intersection of the planes (3b) and (3c) lies entirely within the hyperboloid (3) is not difficult. Choose a view of your display of (3), (3b) and (3c) that illustrates this fact.

End of lab 2.