Math 251:10–12 — Spring 1999
MW6 CHM-106
Prof. Bumby
Lab 3 MAXIMUM AND MINIMUM VALUES

This description of the lab is only for the sections identified in the header. Other sections are expected to use a different version. As in previous labs, you should turn in only the printout of your Maple worksheets with additional material entered using the **text** feature. You can save yourself typing, and have reliable forms of instructions given here, by first copying a prepared Maple worksheet file to your home directory. This is done by going to the course page (if you haven't created a bookmark for it, go to **http://www.math.rutgers.edu/˜bumby** and follow the link to this course), skipping to the bottom of the page, and downloading the "seed file" for lab3. Use only this files, not those designed for other sections that you may find elsewhere. Open this file in *Maple* and modify it as you work through the lab.

In this lab, we use *Maple* to help visualize and compute the maximum and minimum values of a function of two variables. Although the discussion of this topic in Calculus uses *functions*, it is easier to work with *expressions* in Maple. Throughout this worksheet, x and y will be treated as independent variables. All other names will stand for expressions depending on those variables. That is, they are the values of functions of x and y. The functions appearing in this lab, like those appearing elsewhere in the course, are usually polynomials, with a few appearances of exponential or trigonometric functions. The names of these standard functions and the usual notation of algebra allow you to write the expressions for their values to which you apply the rules of calculus.

1. In this problem we wish to find and classify all the critical points of the expression

$$\frac{y^3}{9} + 3x^2y + 9x^2 + y^2 + xy + 9; \tag{A}$$

(This expression is denoted by A in the seed file.)

(a) First obtain a rough idea of what this function looks like, by plotting it over the region $-1.5 \le x \le 1.5$, $-7 \le y \le 1$. The `axes=BOXED` option has been made the default so you can see the values of the function. Note that, in contrast to earlier work, we do not use the `scaling=CONSTRAINED` option, since it is not necessary to compare distances along different axes. Examine the graph for possible critical points, but keep your observations to yourself. The results that you calculate later may turn out to be different from what you expect.

(b) Now use Maple's `diff` and `solve` commands to find all the critical points of the expression (A). A useful convention is to first assign names such as `Ax` and `Ay` to the expressions for $\partial A/\partial x$ and $\partial A/\partial y$ obtained by the `diff` command, i.e.,

```
Ax:=diff(A,x); Ay:=diff(A,y);
```

You could then use the `solve` command to obtain the values of x and y for which both partial derivatives are zero, i.e.,

```
solve({Ax,Ay},{x,y});
```

Note that, if you ask *Maple* to solve an *expression* or a set of expressions, it assumes that you want to solve the equations in which all expressions are equal to zero. It is also a good idea is to name the solutions when they are computed. The solutions obtained from the `solve` command are given to you as an *expression sequence*, i.e., several quantities separated by commas. In this case, the elements in the sequence are sets of assignments of the variables. You can give a name to the whole sequence, e.g.,

```
soln:= solve({Ax,Ay},{x,y});
```

Then you can retrieve individual solutions using the usual *Maple* indexing convention, e.g., `soln[3]` refers to the third one. Hence, to evaluate both `Ax` and `Ay` at `soln[3]`, type

```
subs(soln[3],[Ax,Ay]);
```

The brackets signify a *list* of quantities, and the substituted values will appear in the same order in the output as the expressions did in the input. The "seed file" contains this example, with the warning that it should be removed after you try it. Before you do, make sure you understand why the result is *not* surprising.

(c) Again using Maple's `diff` command, evaluate the second derivatives of (A). The names Axx, Axy, Ayx, and Ayy are a reasonable choice, and are used in the prepared worksheet. As in one of the notations for partial derivatives, the name of the derivative of an expression with respect to a variable is obtained by tacking on the name of the variable to the end of the name of the expression. Using this convention, Axy and Ayx are obtained by different computations, but they should be equal. (To explore this further, see the file **clairaut.mws**, which can be found from the course web page.) The book (see p. 794) classifies critical points using a second derivative test involving a quantity

$$D = \frac{\partial^2 A}{\partial x^2} \cdot \frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 A}{\partial x\, \partial y} \cdot \frac{\partial^2 A}{\partial y\, \partial x}.$$

You are encouraged to *try to* call this expression D. It will lead to an error message, and you should attempt to find an explanation of that message using the help facility. If you succeed, add a comment to your worksheet. Except for this comment, remove all evidence of the error from the final worksheet. Your report should also include **text** describing the information given by this second derivative test. Then compute this expression D and evaluate it, together with anything else that you need to classify the critical point, at each critical point. When you have these values, add a line or two of text giving the type of each critical point.

(d) For each critical point, obtain a plot of the expression (A) in a small region around the critical point, which clearly shows whether it is a local maximum, a local minimum, or a saddle point. Be sure to include a **text** label of each plot giving the coordinates and type of the critical point. Your classification should agree both with the results of (c) and with the appearance of the graph. You will need to experiment to find suitable regions: if the region is too small, the surface will look like a plane; if too large, the behavior at the point may be hidden by the global properties of the graph found in part (a).

2. Now consider the expression

$$y(1 - 10xy)e^{-x^2-y^2}. \qquad (B)$$

You have seen this before in question 9 of Lab 0. Our goal here is to find the absolute minimum and absolute maximum of the expression (B). Before continuing with Maple, don't forget to use the `restart` command; otherwise, previous definitions of your variables may lead to unexpected results.

(a) First obtain a rough idea of what this function looks like, by plotting it over the region $-3 \le x \le 3$, $-3 \le y \le 3$ (this is the same graph that you found in Lab 0). In particular, include **text** describing the value when $|x|$ or $|y|$ is large.

(b) What happens if you use *Maple*'s `diff` and `solve` commands to try to find the critical points of the expression (B)?

Try to find a *Maple* command that gives a *useful* simplification of the derivatives of the expression (B). You may not succeed — *Maple* often seems to refuse to perform what we see as obvious simplifications. Summarize any interesting results of these experiments, but remove the work and output unless it leads to a significant improvement of previous results.

You may also remove the `solve` command and its output from the worksheet, but leave the calculation of the derivatives and **text** describing your adventures with the `solve` command.

(c) Now use the `fsolve` command to find the critical points which lie in the region $-2 \le x \le 2$, $-2 \le y \le 2$. There are exactly four of them, as indicated by the plot in part (a). Also find the values of the function the expression (B) at all of these critical points; be sure that your worksheet makes it clear which values are obtained at which points. **It is not necessary to use a second derivative test in this problem or in part 2d below.**

The `fsolve` command uses an iterative method (such as Newton's method) to find the roots and sometimes the method does not converge to a root. First try

```
fsolve({Bx,By},{x,y});
```

and report your results. The result will probably be unsatisfactory. Consult the help file for the `fsolve` command to find how to restrict the search for a root to a smaller region. The plot obtained in part (a) can help you obtain suitable regions. For example, one of the critical points can be found by using the following form of the `fsolve` command.

```
p1:= fsolve({Bx,By},{x,y}, x =-1..0, y=0..1);
```

Note that as in Problem 1, by assigning the name `p1` to the solution of the `fsolve` command, you can then evaluate the expression (B) at the values of x and y given by Maple in response to this command by typing

```
evalf(subs(p1,B));
```

The command `evalf` is used to convert the result of the `subs` command to a numerical value.

(d) From your answer in (c) and the plot in (a), determine the absolute minimum and absolute maximum values of the expression (B)?

The results of Problem 2 will be used in Problem 3, so do **not** use the `restart` command here.

3. In this problem, we find the absolute minimum and absolute maximum of the expression (B) of Problem 2, but this time restricted to the semicircular region $\mathcal{R}$ given by $y \ge 0$, $x^2 + y^2 \le 3/2$. We know from the general theory that the absolute minimum and maximum of the expression (B) occur either (1) at critical points of the expression (B) which lie in the interior of the region $\mathcal{R}$ or (2) on the boundary of the region $\mathcal{R}$. **As in problem 2, it is not necessary to use a second derivative test in this problem.** The extreme values are found by ruling out all but the values at a small number of points, and those values are computed.

(a) Obtain a plot of the expression (B) with the domain restricted to the region $\mathcal{R}$.

(b) Determine which of the critical points found in Problem 2 lie in the region $\mathcal{R}$ and evaluate the expression (B) at these points. State your conclusions in a **text** section, but leave the calculations supporting your conclusion in the worksheet.

Note that a simple way to check whether the point given by `p1` is inside the circle $x^2 + y^2 = 3/2$ is use the `subs` command to evaluate the expression $x^2 + y^2 - 3/2$ at `p1`. If this value is less than zero **and** if the y component of `p1` is ≥ 0, then the point `p1` lies in the region $\mathcal{R}$.

(c) The boundary of $\mathcal{R}$ consists of two pieces. The first is the line segment

$$y = 0 \qquad -\sqrt{\frac{3}{2}} \le x \le \sqrt{\frac{3}{2}}.$$

What is the value of the expression (B) on this line segment? Give answer as **text**. Note that you don't need Maple to find it.

(d) The second piece of the boundary of $\mathcal{R}$ is the semicircle

$$-\sqrt{\frac{3}{2}} \le x \le \sqrt{\frac{3}{2}} \qquad y = \sqrt{\frac{3}{2} - x^2}.$$

This curve can be parameterized by writing $x = \sqrt{3/2}\cos t$ and $y = \sqrt{3/2}\sin t$ for $0 \le t \le \pi$. Create a new expression (C) using the command

```
C:=subs(x=sqrt(3/2)*cos(t),y=sqrt(3/2)*sin(t),B);
```

and find the maximum and minimum of (C) over the interval $0 \le t \le \pi$.

Then, use the `plot` command to obtain a plot of the expression (C) over the interval $0 \le t \le \pi$. From the plot you should be able to see roughly where the maximum or minimum occur. In particular, you can see whether they occur at one of the end points of the interval or at a place where the derivative is zero. Then, you can apply `fsolve` on a restricted interval to locate the critical points of the expression (C) corresponding to the minimum and maximum.

A graph of the derivative of expression (C) will identify *three* points where that function is zero, corresponding to three critical points of (C). In addition to the locations of the maximum and minimum on the boundary, there is a local maximum near $t = .0333767$. A simple equation for $\cot t$ can be found although *Maple* appears to seek only more complicated expressions for the solution.

(e) Now compare the values of the expression (B) at the critical points inside $\mathcal{R}$ (from part (b)), on the line segment $y = 0$ (from part (c)), and on the semicircle (from part (d)). Clearly state the maximum and minimum values of the expression (B) over the region $\mathcal{R}$ and the coordinates of the points at which they occur. Check that your conclusion agrees with the plot found in (a).