This may not have been done correctly last time.

Exercise 11.1.9 Are the points

$$P(1, 2, 3) \quad Q(0, 3, 7) \quad R(3, 5, 11)$$

collinear?

Solution. Find distances

$$|PQ| = \sqrt{1^2 + 1^2 + 4^2} = \sqrt{18}$$

$$|PR| = \sqrt{2^2 + 3^2 + 8^2} = \sqrt{77}$$

$$|QR| = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{29}$$

Is the largest equal to the sum of the other two?

$$\sqrt{18} + \sqrt{29} \quad ? \quad \sqrt{77}$$

Square

$$47 + 2\sqrt{522} \quad ? \quad 77$$

Subtract 47, divide by 2

$$\sqrt{522} \quad ? \quad 15$$

Square

$$522 \quad ? \quad 225$$

NO.

In elementary geometry, you learn that a triangle is determined by the lengths of its sides. In trigonometry, the law of cosines is derived to show that the angle θ between sides a and b and opposite side c is determined by

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

If we use vectors **a** and **b** for the vectors pointing away from the vertex C, then a vector along the third side is $\mathbf{a} - \mathbf{b}$, and

$$c^2 = |\mathbf{a} - \mathbf{b}|^2 .$$

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\begin{aligned} c^2 &= (a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2 \\ &= a_1^2 - 2a_1b_1 + b_1^2 \\ &+ a_2^2 - 2a_2b_2 + b_2^2 \\ &+ a_3^2 - 2a_3b_3 + b_3^2 \\ &= a^2 + \qquad\qquad + b^2 \end{aligned}$$

Comparing this with the law of cosines, gives

$$ab\cos\theta = a_1b_1 + a_2b_2 + a_3b_3.$$

The expression on the right is taken as the definition of the **dot product** $\mathbf{a} \cdot \mathbf{b}$, and the geometric interpretation is a theorem. The form of the definition shows that the dot product is *linear* in each factor. A special case of the definition shows that $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$, and the Pythagorean theorem shows that nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. It is convenient to extend the use of the word "perpendicular" so the a zero vector is considered perpendicular to all vectors.

Projections and components. We have the formulas

$$\begin{aligned} \operatorname{comp}_{\mathbf{a}} \mathbf{b} &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \\ \operatorname{proj}_{\mathbf{a}} \mathbf{b} &= \left(\operatorname{comp}_{\mathbf{a}} \mathbf{b}\right) \frac{\mathbf{a}}{|\mathbf{a}|} \\ &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} \end{aligned}$$

The cross product. The formula summarized by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \qquad (*)$$

clearly gives a vector perpendicular to

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle \text{ and } \mathbf{b} = \langle b_1, b_2, b_3 \rangle,$$

using a systematic formula. It is zero when $\mathbf{a} \parallel \mathbf{b}$, and only in this case. Direct computation shows that the length of this vector is the product of the lengths of **a** and **b** and the sine of the angle between them. This shows that the length is the area of the parallelogram spanned by **a** and **b**. The only surprise is that interchanging the roles of **a** and **b** sends the product into its negative. Formula $(*)$ defines the cross product $\mathbf{a} \times \mathbf{b}$. In addition to giving directions perpendicular to two given directions and finding areas, it gives a computational way to describe orientation in space — the distinction between *left* and *right*.

Equation of line. A line in space is determined by a point $P_0(x_0, y_0, z_0)$ and a direction $\mathbf{v} = \langle a, b, c \rangle$. Once you have the six numbers, all that is needed to get the equation is to write them in the right places. Things are slightly complicated by three ways of writing the equation being in common use. The keywords identifying these forms must be known.

Vector form. The form closest to the geometry of the line uses the idea that vectors in the same direction are related by being scalar multiples of one another. Thus, the vector $\overrightarrow{P_0 P}$ from the given point P_0 to the variable point $P(x, y, z)$ should be a multiple of the vector $\mathbf{v}$. The scalar multiple is usually denoted by t, although different letters should be used for different lines in the same problem. Changing the parentheses around the quantities in P_0 and P to angle brackets transforms them into the vectors $\mathbf{r}_0 = \overrightarrow{OP_0}$ and $\mathbf{r} = \overrightarrow{OP}$, and the vector equation becomes

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}.$$

Parametric form. The vector equation is an abbreviation for the three scalar equations obtained by equating the components. For the data given above, this is

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Symmetric form. Each of these equations can be solved for the parameter t. The point P lies on the line if you always get the same answer. This puts the equation in a form that emphasizes a test for whether P lies on the line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

This may even be used if some of the coefficients a, b or c are zero. The other forms show that this should be interpreted as requiring that a zero denominator requires the corresponding numerator to be zero.

How else could a line be given? Another way to give a line is by specifying two points P_0 and $P_1(x_1, y_1, z_1)$ on the line. This is immediately converted into the original form by taking $\mathbf{v} = \overrightarrow{P_0 P_1}$.

A line may also be given as an intersection of two planes. We will return to this after discussing how planes are given.

Parallel lines; intersections of lines; skew lines. Lines are said to be parallel if they have the same direction. Thus, $P_0 + t\mathbf{v}_0$ is parallel to $P_1 + u\mathbf{v}_1$ if there is a constant λ such that $\mathbf{v}_0 = \lambda\mathbf{v}_1$. The *inclusive* form of this definition means that *identical* lines are considered to be parallel. In our example, the lines are identical if $\overrightarrow{P_0 P_1}$, $\mathbf{v}_0$, and $\mathbf{v}_1$ all have the same direction. Although the same line may have two equations that appear different, it is easy to check whether different equations represent the same line.

Two lines intersect if you can find t and u so that

$$P_0 + t\mathbf{v}_0 = P_1 + u\mathbf{v}_1.$$

In scalar form, this gives three equations relating the two variables t and u. This suggests that pairs of lines do not usually intersect. A simple example is

$$t\langle 1, 0, 0\rangle \text{ and } (0, 0, 1) + u\langle 0, 1, 0\rangle.$$

Lines that are not parallel and do not intersect are called *skew*.

The equation of a plane. A single equation

$$ax + by + cz = d \qquad (*)$$

defines a plane. To see this, suppose you have one solution (x_0, y_0, z_0) of (*), which could be obtained by setting $x = x_0$ and $y = y_0$ arbitrarily and solving $(*)$ for z to find z_0. Then $(*)$ is equivalent to

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \qquad (**)$$

This says that the general vector in the plane $(**)$ $\overrightarrow{P_0P}$ is perpendicular to $\mathbf{n} = \langle a, b, c\rangle$. This leads to a geometric way of recognizing the solutions of $(*)$ as a plane. It also says that the equation can be written if one knows a point in the plane and a direction perpendicular to the plane.

The plane through three points. A common way to describe a plane is to give three points P_0, P_1, P_2 in the plane. Then P_0P_1 and P_0P_2 are two directions in the plane and $P_0P_1 \times P_0P_2$ is perpendicular to the plane. This, together with the point P_0 in the plane gives the information needed to write the equation.

Intersection of line and plane. In general, an equation gives a condition on the coordinates (x, y, z) of a general point for it to lie in a set. The intersection of a line, given in parametric form, and a plane, given by an equation, is found by substituting the parametric description of the points on the line into the equation for the plane. This gives a single equation in the parameter. Solve this equation and use that value in the description of the line to find the coordinates of the intersection.

Intersection of two planes. The easiest way to find the line of intersection of two planes is to use the geometric interpretation of the equation of a plane. The coefficients give a direction perpendicular to the plane, i.e. perpendicular to each direction in the plane. The direction of the line of intersection is thus perpendicular to the two vectors giving the directions of the planes, so its direction is given by the cross product. It remains to find a point on the line. This can be done by choosing the z-coordinate arbitrarily ($z = 0$ is a good choice). The equations of the planes then give two equations in x and y that are easy to solve simultaneously.

Vector functions. We consider functions from $\mathbb{R}$ to $\mathbb{R}^3$. The appropriate notion of continuity of such functions is that each component be continuous. Similarly, to differentiate such a function, use the derivatives of the components as the components of a vector. These properties are not arbitrary: they can be proved from a general definition of limit. If the values of a vector function are plotted, one gets a **space curve**. The simplest example is a linear function, whose graph is a line. Several other examples appear in the text. An interesting example is the **helix**, given by $\langle \cos t, \sin t, t \rangle$. On the surface where $x^2 + y^2 = 1$, this curve moves evenly around the cylinder and up the axis.

Tangent vector and arc length. If the vector function $\mathbf{r}(t)$ is thought of as giving position as a function of time, then its derivative $\mathbf{r}'(t)$ gives **velocity**. The length of $\mathbf{r}'(t)$ measures the **speed** and we shall see that the distance travelled along the curve is the integral of speed. A unit vector in the direction of $\mathbf{r}'(t)$ is called the **unit tangent vector** and denoted $\mathbf{T}$. Since it is obtained by scaling the vector between nearby points on the curve, it measures the direction of motion as a function of t. A line in this direction is defined to be the tangent line to the curve. You should look at some examples to convince yourself that this is a reasonable definition. If a different parameter u is used to describe the curve, with t being an increasing function of u, $d\mathbf{r}/du = (dt/du)(d\mathbf{r}/dt)$. The first factor is a scalar, so it does not affect T. The usual rules of calculus for sums and products are easily proved for derivatives of vectors. One consequence of this is that if $\mathbf{a}(t)$ is of constant length, so that $\mathbf{a}(t) \cdot \mathbf{a}(t)$ is a constant function, then

$$0 = \mathbf{a}(t) \cdot \mathbf{a}'(t) + \mathbf{a}'(t) \cdot \mathbf{a}(t) = 2\mathbf{a}(t) \cdot \mathbf{a}'(t)$$

so that $\mathbf{a}'(t)$ is always perpendicular to $\mathbf{a}(t)$.

Normals and curvature, From the last result, we get that $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$. The direction of $\mathbf{T}'(t)$ is called the **principal normal** and denoted **N**. Changing the parameter multiplies the derivative of **T** by a scalar (positive if the parameters are increasing functions of one another), so **N** is independent of the parameterization. If you take *arc length* as the parameter, then the magnitude of the derivative is also significant. This value is called **curvature**, and denoted κ, here described by **definition (8)**. Finally, it is not actually necessary to construct this parameterization, since the value at any point can be found from the chain rule. This gives **formula (9)**, which we use in the example below. However, this gives all geometric features as functions of the original parameter.

The main example. If $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$, describing a circle of radius a in the xy plane, $\mathbf{r}'(t) = \langle -a\sin t, a\cos t\rangle$, and we can *see* its length and direction: $ds/dt = a$ and $\mathbf{T} = \langle -\sin t, \cos t\rangle$. Then

$$\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{ds/dt} = \frac{\langle -\cos t, -\sin t\rangle}{a}$$

in this case. Geometrically, we see that $\mathbf{N}$ is a unit vector pointing towards the center of the circle, and $\kappa = 1/a$.

Another curvature formula. Another formula for curvature differentiates $\mathbf{r}'(t)$ directly without rescaling it into $\mathbf{T}(t)$. This appears as **Theorem (10)**.

$$\kappa(t) = \frac{\left|\mathbf{r}'(t) \times \mathbf{r}''(t)\right|}{|\mathbf{r}'(t)|^3}.$$

The proof is direct. The product rule gives

$$\mathbf{r}''(t) = \frac{d^2 s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'.$$

Since $\mathbf{T} \parallel \mathbf{r}'(t)$, the first term contributes nothing when you take the cross product with $\mathbf{r}'(t)$, and

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\frac{ds}{dt}\right)^2 \mathbf{T} \times \mathbf{T}'.$$

Finally, $\mathbf{T} \perp \mathbf{T}'$, so the length of the cross product is the product of the lengths, and $\mathbf{T}$ is a unit vector, while the length of $\mathbf{T}'$ (from (9)) is $\kappa(ds/dt)$. This formula is most useful if ds/dt is complicated.

Motion in space. These formulas are used in mechanics. If $\mathbf{r}(t)$ represents the **position** of a body as a function of time, then $\mathbf{r}'(t)$ is **velocity** and $\mathbf{r}''(t)$ is **acceleration**. The key idea in Newton's explanation of motion was that motion represented the effect of **forces** and **force** is **mass** times **acceleration**. All of these quantities except **mass** are vectors; **mass** is a scalar that is constant for ordinary objects. If you observe the position function $\mathbf{r}(t)$, you determine the acceleration $\mathbf{r}''(t)$ and use that to help identify the force.

In Section 11.8, the formula

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'.$$

was derived. We also have $\mathbf{T}' = \kappa(ds/dt)\mathbf{N}$, so

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$

Writing v in place of ds/dt gives formula (7) of Section 11.9. The quantity v represent the **speed** of the object. The vectors $\mathbf{T}$ and $\mathbf{N}$ are perpendicular unit vectors that are part of a coordinate system that moves with the object. In particular $\mathbf{T}$ is "straight ahead". In this coordinate system, the first part of the expression for $\mathbf{r}''(t)$ describes the part of the acceleration (and, hence, of the force) that leads to a change of speed, while the second part describes the part of the acceleration that leads to a change of direction.

The textbook leads you through the use of this approach to derive Kepler's observations about planetary motion from Newton's hypothesis about the nature of the gravitational force. While exam questions in this course will concentrate on more primitive topics, this example establishes the historical importance of this use of vectors.

Exercise 11.9.27. This was started at end of the last lecture before exam, but not done in an efficient manner. The following treatment is slightly different from the method developed in the text, and leads to an *algorithm* for curvature that is not easily summarized in a formula. We are looking for a calculation of the scalar coefficients of **T** and **N** in

$$\mathbf{r}''(t) = \frac{d^2s}{dt^2}\,\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N},$$

when

$$\mathbf{r}(t) = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j}.$$

In stating the problem, the word "components" is used to refer to these coefficients, which should suggest

$$\begin{aligned}\frac{d^2s}{dt^2} &= \operatorname{comp}_{\mathbf{T}}\mathbf{r}''(t)\\ &= \operatorname{comp}_{\mathbf{r}'(t)}\mathbf{r}''(t)\end{aligned}$$

since $\mathbf{T} \parallel \mathbf{r}'(t)$

$$= \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

In this case,

$$\begin{aligned}
\mathbf{r}(t) &= (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j} \\
\mathbf{r}'(t) &= (1 - \cos t)\,\mathbf{i} + \sin t\,\mathbf{j} \\
\mathbf{r}''(t) &= \sin t\,\mathbf{i} + \cos t\,\mathbf{j} \\
\left|\mathbf{r}'(t)\right|^2 &= 2 - 2\cos(t) \\
\mathbf{r}'(t) \cdot \mathbf{r}''(t) &= \sin t
\end{aligned}$$

Thus,

$$\frac{d^2 s}{dt^2} = \sin t \big(2 - 2\cos(t) \big)^{-1/2}.$$

Also,

$$\begin{aligned}
\operatorname{proj}_{\mathbf{T}} \mathbf{r}''(t) &= \operatorname{proj}_{\mathbf{r}'(t)} \mathbf{r}''(t) \\
&= \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|^2} \mathbf{r}'(t) \\
&= \frac{\sin t}{2 - 2\cos(t)} \big((1 - \cos t)\,\mathbf{i} + \sin t\,\mathbf{j} \big)
\end{aligned}$$

If this is subtracted from $\mathbf{r}''(t)$, one gets a vector perpendicular to $\mathbf{T}$ (a fact which can be used to derive these formulas easily). Simplifying this gives

$$\kappa\left(\frac{ds}{dt}\right)^2\mathbf{N} = \frac{1}{2}\sin t\,\mathbf{i} - \frac{1}{2}(1-\cos t)\,\mathbf{j}.$$

This vector is *clearly* perpendicular to $\mathbf{T}$, and finding its length gives

$$\kappa\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}\big(2-2\cos(t)\big)^{1/2}.$$