Math 251:10–12 — Spring 1999
MW6 CHM-106
Prof. Bumby

Fifth set of workshop problems — Tangent Plane, Chain Rule, Gradients
Textbook Sections 12.5 – 12.6

1. Let $z = f(x, y)$, $x = r\cos\theta$, and $y = r\sin\theta$.

(a) Use the chain rule to find $\partial z/\partial r$ and $\partial z/\partial\theta$ in terms of r, θ, $\partial z/\partial x$ and $\partial z/\partial y$.

(b) Use repeated application of the formulas in (a) to show that for general f,

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial^2 z}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 z}{\partial x\partial y}\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2}\sin^2\theta,$$

$$\frac{\partial^2 z}{\partial\theta^2} = \frac{\partial^2 z}{\partial x^2}r^2\sin^2\theta - 2\frac{\partial^2 z}{\partial x\partial y}r^2\sin\theta\cos\theta + \frac{\partial^2 z}{\partial y^2}r^2\cos^2\theta - \frac{\partial z}{\partial x}r\cos\theta - \frac{\partial z}{\partial y}r\sin\theta.$$

(c) Use the result of (b) to show that

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}.$$

(d) For the particular function

$$f(x, y) = x^2 + y^2 + 2xy,$$

simplifying $f(r\cos\theta, r\sin\theta)$ gives

$$z = r^2\big(1 + \sin(2\theta)\big).$$

Verify this simplification and use this expression to directly calculate the partial derivatives of z with respect to r and θ appearing in (a), (b) and (c). Then, compare these results to the formulas in terms of r, θ, $\partial z/\partial x$ and $\partial z/\partial y$ obtained from the chain rule.

2. Show that the product of the x, y, and z intercepts of any tangent plane to the surface $xyz = c^3$ is a constant.

3. Suppose that at the point $(1, 2)$, the directional derivative of the function $z = f(x, y)$ in the direction of $2\mathbf{i} + 3\mathbf{j}$ is equal to 5 and the directional derivative of $z = f(x, y)$ in the direction of $-4\mathbf{i} + \mathbf{j}$ is equal to 4.

(a) Determine the gradient of f at the point $(1, 2)$.

(b) What is the maximum rate of change of the function f at the point $(1, 2)$?