

Math 251:10-12 — Spring 1999

MW6 CHM-106

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**Tenth set of workshop problems — change of coordinates, Stokes' Theorem, Divergence Theorem.
Textbook Sections 13.9, 14.8, 14.9**

1. Consider the integral $\iint_{\mathcal{R}} \cos((x+y)^2) dA$, where \mathcal{R} is the triangular region with vertices $(0, 0)$, $(\pi, 0)$, and $(0, \pi)$.

(a) Write this integral as an iterated integral, integrating first with respect to x and then with respect to y .

(b) Reverse the order of integration for the integral in (a).

(c) Consider the change of variable $u = x + y$, $v = y$. Sketch the region \mathcal{S} in the uv plane corresponding to the region \mathcal{R} in the xy plane.

(d) Write the integral as an iterated integral in u and v , integrating first with respect to u and then with respect to v .

(e) Reverse the order of integration for the integral in (d).

(f) Evaluate $\iint_{\mathcal{R}} \cos((x+y)^2) dA$ by using *one* of the four formulas you have found.

2. Let \mathcal{S} be a smooth surface with whose boundary is a simple closed curve \mathcal{C} . Assume that f and g are functions with continuous second-order partial derivatives and \mathbf{F} is a vector field with continuous partial derivatives..

(a) Show that

$$\nabla \times (f \mathbf{F}) = f (\nabla \times \mathbf{F}) + (\nabla f) \times \mathbf{F}.$$

(b) Use Stokes' Theorem and (a) to show that

$$\int_{\mathcal{C}} (f \nabla g) \cdot d\mathbf{r} = \iint_{\mathcal{S}} (\nabla f \times \nabla g) \cdot \mathbf{n} dS$$

(c) Use (b) to show

$$\int_{\mathcal{C}} (f \nabla f) \cdot d\mathbf{r} = 0.$$

(d) Show

$$\int_{\mathcal{C}} (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0.$$

... more

3. Let \mathcal{E} be a solid region with boundary surface \mathcal{S} and outward normal vector \mathbf{n} . Let f be a scalar function and \mathbf{G} a vector field with continuous second order partial derivatives defined in an open region containing \mathcal{E} .

(a) Show that

$$\nabla \cdot (f \mathbf{F}) = f (\nabla \cdot \mathbf{F}) + (\nabla f) \cdot \mathbf{F}.$$

(b) Use the Divergence Theorem and (a) to prove

$$\iint_{\mathcal{S}} f \mathbf{G} \cdot \mathbf{n} dS = \iiint_{\mathcal{E}} f \nabla \cdot \mathbf{G} + \nabla f \cdot \mathbf{G} dV.$$

(c) Use (b) with $\mathbf{G} = \nabla g$ to show

$$\iint_{\mathcal{S}} f \nabla g \cdot \mathbf{n} dS = \iiint_{\mathcal{E}} f \nabla^2 g + \nabla f \cdot \nabla g dV,$$

where ∇^2 is the **Laplace operator** defined on p. 907 of the textbook.

(d) Show

$$\iint_{\mathcal{S}} g \nabla f \cdot \mathbf{n} dS = \iiint_{\mathcal{E}} g \nabla^2 f + \nabla f \cdot \nabla g dV.$$

(e) Use (c) and (d) to show

$$\iint_{\mathcal{S}} (f \nabla g - g \nabla f) \cdot \mathbf{n} dS = \iiint_{\mathcal{E}} f \nabla^2 g - g \nabla^2 f dV.$$

Remarks. The notation $\mathbf{G} \cdot \mathbf{n} dS$ has been used to describe the surface integral of the vector field \mathbf{G} that appears in Stokes' Theorem and the divergence theorem. In case $\mathbf{G} = \nabla g$, $\nabla g \cdot \mathbf{n}$ is the derivative of the function g with respect to the outward unit normal vector \mathbf{n} , and the surface integrals of this problem may be interpreted as integrals with respect to surface *area* whose integrand contains geometrically defined directional derivatives. In this form, the results of (c) and (e) are known as Green's identities.

The cross product, and the related operation of the curl of a vector field, uses special properties of three dimensional Euclidean space in its definition. A more general approach, with new notation, is needed to define oriented integrals on h dimensional surfaces in a k dimensional space and relate them by a form of Stokes' theorem.

This general theory is not needed to extend the proof of the divergence theorem to domains in \mathbb{R}^k for arbitrary k , as long as one has a clear idea of $k - 1$ dimensional "area" on the boundary. In particular, Green's identities are not restricted to three dimensional space, and this problem is an outline of the general proof.