Math 252:01 — Spring 2002
MTh3 SEC-212
Prof. Bumby

Project 1: Three differential equations

Project should be handed in by Monday, February 25. Grade will be based on content as well as on clarity and neatness of presentation. The use of a computer is neither required nor prohibited.

Piecewise continuous problems. The basic existence and uniqueness theorems, as stated in the textbook, guarantee that the initial-value problem

$$y' = f(t, y)$$
$$y(t_0) = y_0$$

has a unique solution (at least for some small time interval), provided that f and $\frac{\partial f}{\partial y}$ are *continuous* as functions of (t, x). However, practical problems are often described with *abrupt* changes. For example, the *On-off voltage source* example in the textbook (pp. 45–46) describes the action of a switch as an instantaneous change from a voltage of K to zero voltage. Reality is more complicated, but this is what the design aims to achieve. On the usual scale, the everyday world behaves *smoothly*, that is the derivatives needed to state the laws of mechanics exist and are continuous. The first equation in this project examines an extension of these methods to cases in which smoothness *appears to be* violated. In the second equation, the discontinuities are smoothed over, but this does not necessarily lead to a simpler analysis. The third equation has some properties in common with the other two, and is easy to solve, but its solutions may not resemble the solutions of the other equations.

The first equation. The equation to be studied here was already mentioned in lecture. It is

$$\frac{dy}{dt} = \begin{cases} 1 & \text{if } y > t^2 \\ 0 & \text{if } y < t^2 \end{cases}. \tag{F}$$

Let the right side of (F) be denoted $f(t, y)$. Note that $f(t, y)$ is undefined on the curve $\mathcal{C}$ where $y = t^2$.

On either side of the curve $\mathcal{C}$, the equation has a form to which the theorems of Section 1.5 apply. We will use the word **component** to refer to one of these sets. Moreover, as long as you only see the component where $y > t^2$, the equation looks like $dy/dt = 1$, and you **have a formula** for all solutions. Similarly, you have a formula for all solutions in the component where $y < t^2$.

First Task. Begin by stating the formulas for these solutions and giving a **picture** of the solutions (don't just *sketch* the solutions; use an appropriate tool to show them accurately). Use the formulas for the solutions to show that the general point on $\mathcal{C}$ lies on a unique solution in each of the components by writing a formula for the solution in each component through a point (t_0, y_0) with $y_0 = t_0^2$. Although these solutions are well-behaved within each component, they may not fit together to give a solution $y(t)$ that spans the two components. Classify points of $\mathcal{C}$ by whether the solutions in the two components combine to give a unique continuous function $y(t)$ defined on an interval around t_0 that satisfies (F) at points not on $\mathcal{C}$.

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A continuous refinement. Now consider

$$\frac{dy}{dt} = \begin{cases} 1 & \text{if } y \ge t^2 + 0.01 \\ 0 & \text{if } y \le t^2 \\ \sin^2\left(50\pi(y - t^2)\right) & \text{if } t^2 \le y \le t^2 + 0.01 \end{cases} . \qquad (G)$$

Let the right side of (G) be denoted $g(t, y)$. This agrees with the original equation except on a thin band between $\mathcal{C}$ and the curve $\mathcal{C}'$ where $y = t^2 + 0.01$.

Second task. Verify that the $g(t, y)$ and its partial derivative with respect to y, which we will denote g_y, are continuous on the whole (t, y)-plane. A full ϵ, δ proof is not required, but you should show that the expressions for both g and g_y on either side of $\mathcal{C}$ reduce to the same value, and similarly on the curve $\mathcal{C}'$. The principle here is that algebraic expressions were shown to be continuous in the course of finding differentiation formulas, so all limits involving those formulas such expressions are found by evaluation. What you **must do carefully** is to determine the expressions for g_y.

Is there a visible difference? At normal resolution, there is no visible difference between the slope fields of (F) and (G). However, you should have found some points (t_0, y_0) on $\mathcal{C}$ for which the solutions of (F) that approach (t_0, y_0) are only defined for $t > t_0$. By contrast, the existence and uniqueness theorem says that the solutions to (G) through such points must be defined for all t in an interval around t_0.

In fact, each choice of an initial value (t_0, y_0) (anywhere in the plane) defines solution of (G) that gives y as a function of t for all t. This follows from the fact that

$$0 \le g(t, y) \le 1, \qquad (B)$$

so that the approximations given by Euler's method have y between y_0 and $y_0 + t - t_0$.

Third task. Use (B) to find bounds on where the solution to (G) through $(-1, 1)$ on $\mathcal{C}$ first meets $\mathcal{C}'$. From this point, the solution will be above $\mathcal{C}'$, and hence will agree with a solution of (F) until it meets $\mathcal{C}'$ for a second time. Then (B) can be applied again to bound the second point where this solution meets $\mathcal{C}$. You should find that the solution to (G) through $(-1, 1)$ is approximated by a unique continuous function that is a solution to (F) at points not on $\mathcal{C}$.

Fourth task. Now consider the point $(-0.05, 0.0025)$. First, note that (B) does not guarantee that the solution through this point will cross $\mathcal{C}'$. Use a numerical method to investigate whether the solution through this points crosses $\mathcal{C}'$ or remains below it. Since a large number of steps are required for acceptable accuracy, this should be done with a programmable calculator or computer. Your results may be presented as a graph or as a small sample of the computed values. Describe of the method used, including your reason for trusting the computed results.

A related equation. Consider the equation

$$\frac{dy}{dt} = y - t^2. \qquad (H)$$

As before, let $h(t, y)$ denote the right side of (H). The curve $\mathcal{C}$ plays an important role in this equation since $h(t, y) = 0$ on $\mathcal{C}$, and $h(t, y) > 0$ above $\mathcal{C}$.

Equation (H) is linear, so it is possible to find an analytic solution that can be used to study properties of the solutions of the equation.

Fifth task. Find a general solution of (H) and use it to classify solutions in terms of the number of times that they cross $\mathcal{C}$.

End of Project 1.